



Subspace structure of some operator and Banach spaces

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Abstract: We construct a family of separable Hilbertian operator spaces, such that the relation of complete isomorphism between the subspaces of each member of this family is complete \mathbf{K}_σ . We also investigate some interesting properties of completely unconditional bases of the spaces from this family. In the Banach space setting, we construct a space for which the relation of isometry of subspaces is equivalent to equality of real numbers.

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1 Introduction and main results

Recently, there has been much progress in describing the complexity of various relations between subspaces of a given separable Banach space. The reader is referred to [4, 5, 6] for the known results on the relations of isomorphism, biembeddability, and more. Isometry and local equivalence (finite representability) are handled in [16] and [7], respectively.

In this paper, we consider an operator space analogue of this problem (see Section 2 for a brief introduction into operator spaces). The Effros-Borel structure on the set $\mathbf{S}(Z)$ of infinite-dimensional subspaces of a separable operator space Z is defined in the same way as for Banach spaces, see e.g. [11, Chapter 12], or [6]. Reasoning as in Section 2 of [6], we show that the relations of complete isomorphism, complete biembeddability, and such, defined on $\mathbf{S}(Z)$, are analytic equivalence relations.

In the commutative space setting, the famous Gowers-Komorowski-Tomczak Theorem (see e.g. [25]) shows that any separable Banach space, isomorphic to all of its infinite dimensional subspaces, must be isomorphic to ℓ_2 . The space ℓ_2 can be equipped with uncountably many 1-homogeneous operator space structures (an operator space X is called 1-homogeneous if the equality $\|u\| = \|u\|_{cb}$ holds for any $u \in B(X)$). For

instance, the row and column spaces \mathbf{R} and \mathbf{C} , as well as their complex interpolation spaces $(\mathbf{R}, \mathbf{C})_\theta$, have this property. These, and other, examples of homogeneous Hilbertian spaces can be found, for instance, in [21, 22] (an operator space is called *Hilbertian* if it is isomorphic to a Hilbert space). Clearly, any 1-homogeneous operator space, having ℓ_2 as its underlying Banach space, is completely isometric to all of its infinite dimensional subspaces. For such spaces, the relation of complete isomorphism between infinite dimensional subspaces is trivial.

It is not known how “simple” the relation of complete isomorphism between subspaces of X may get if X is not homogeneous (or not isomorphic to a Hilbert space). Our main result provides a partial answer to this question.

Theorem 1.1 *There exists a family \mathfrak{F} of operator spaces, such that, for any operator space $X \in \mathfrak{F}$, the following is true:*

- (1) X is isometric to ℓ_2 .
- (2) *The relation of complete isomorphism and complete biembeddability on $\mathbf{S}(X)$ are Borel bireducible to the complete \mathbf{K}_σ relation.*

The family \mathfrak{F} contains a continuum of operator spaces, not completely isomorphic to each other.

It is possible to prove that the relation of complete isometry on $\mathbf{S}(X)$ ($X \in \mathfrak{F}$) is Borel bireducible to the equality on \mathbb{R} . The proof proceeds along the same lines as Theorem 6.1, but is exceedingly technical, and not very illuminating. We therefore omit it, and present a related Theorem 1.4 instead.

Each space from \mathfrak{F} has its *canonical basis* (defined in Section 4), which is 1-completely unconditional. It turns out these bases have interesting properties of their own.

Theorem 1.2 *Any subspace Y of an operator space X from the family \mathfrak{F} has 1-completely unconditional canonical basis. Any C -completely unconditional basis in such a Y is $\phi(C)$ -equivalent (up to a permutation) to the canonical basis of Y , with $\phi(C)$ polynomial in C .*

For certain spaces X , a stronger result holds.

Theorem 1.3 *For any $a > 1$ there exists an operator space X , belonging to the family \mathfrak{F} , such that the canonical basis in any subspace of X is a -equivalent to a subsequence of the canonical basis of X . Consequently, every C -completely unconditional basic sequence in X is $\phi(C)$ -equivalent (up to a permutation) to a subsequence of the canonical basis, with $\phi(C)$ polynomial in C .*

Note that homogeneous Hilbertian operator spaces satisfy the conditions of the three theorems above. The spaces from the family \mathfrak{F} are not homogeneous, yet, by Theorems 1.2 and 1.3, they share certain essential properties of homogeneous spaces.

As part of our motivation lies in the field of Banach spaces, we mention a few classical results related to Theorems 1.1, 1.2, and 1.3.

No “commutative” counterpart of Theorem 1.1 has yet been obtained: for a Banach space E , very little is known about the “upper” estimates on the complexity of the isomorphism relation on $\mathbf{S}(E)$. It is possible that the relation of isomorphism on $\mathbf{S}(E)$ is complete analytic whenever E is a separable Banach space, not isomorphic to ℓ_2 .

Theorem 1.2 shows (among other things) that every subspace of X has an unconditional basis. In the commutative setting, ℓ_2 is the known only separable Banach space with the property that each of its separable subspaces has an unconditional basis. It is a long-standing open question (see e.g. [8, p. 203]) whether there exist other spaces sharing the same property. If a separable E has this property, it must be “close to” ℓ_2 . First, any subspace of E has the Approximation Property, therefore, by [15, Theorem 1.g.6], E has to have type $2 - \varepsilon$ and cotype $2 + \varepsilon$ for any $\varepsilon > 0$. Furthermore, by [20, Theorem 10.13], E has weak cotype 2. By [12], E is ℓ_2 -saturated. Finally, if $E = \ell_2(F)$ for some Banach space F , then E is isomorphic to a Hilbert space [13].

Searching for a “commutative” analogue of Theorem 1.3, we strengthen the question posed in the previous paragraph: suppose a Banach space E has an unconditional basis (e_i) , such that every subspace of E has an unconditional basis, equivalent to a subsequence of (e_i) . Must E be isomorphic to ℓ_2 ?

Finally, in the Banach space setting, we construct a space E , not isometric to ℓ_2 , but whose subspace structure is “very simple.”

Theorem 1.4 *The following equivalence relations between infinite dimensional subspaces of $\mathbb{R} \oplus_1 \ell_2$ (or, in the complex case, $\mathbb{C} \oplus_1 \ell_2$) are Borel bireducible to $(\mathbb{R}, =)$: (i) isometry, (ii) having Banach-Mazur distance 1; (iii) isometric bi-embeddability; (iv) almost isometric bi-embeddability.*

Recall that the Banach-Mazur distance between Y and Z is defined as $d(Y, Z) = \inf\{\|u\|\|u^{-1}\| : u \in B(Y, Z)\}$. Y is called *almost isometrically embeddable* into Z if for every $\varepsilon > 0$ there exists $Y_\varepsilon \hookrightarrow Z$ s.t. $d(Y, Y_\varepsilon) < 1 + \varepsilon$ ($d(\cdot, \cdot)$ denotes the Banach-Mazur distance between spaces).

The rest of the paper is organized as follows: Section 2 provides a brief introduction into operator spaces and c.b. maps. In Section 3 we construct, for each contraction

$A \in B(\ell_2)$, a subspace $X(A)$ of $\mathbf{R} \oplus \mathbf{C}$, and study the properties of the spaces $X(A)$. Further investigation is carried out in Section 4, where we concentrate on the case when A is compact. Unconditional bases in the spaces $X(A)$ are described in Section 5. In Section 6 we show that, for the “right” compact contractions A , the relations of complete isomorphism and complete biembeddability on $\mathbf{S}(X(A))$ are Borel bireducible to the complete \mathbf{K}_σ relation. Having gathered all the preliminary results, we prove Theorems 1.1, 1.2, and 1.3 in Section 7. Finally, in Section 8 we prove Theorem 1.4.

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2 Introduction into operator spaces

As our paper deals primarily with operator spaces, we are compelled to recall some basic definitions and facts about the topic. An interested reader is referred to [3, 19, 22] for more information. A (*concrete*) operator space X is, for us, just a closed subspace of $B(H)$ (H is a Hilbert space). If X and Y are operator spaces, embedded into $B(H)$ and $B(K)$ respectively, we defined the *minimal tensor product* of X and Y (denoted simply by $X \otimes Y$) as the closure of the algebraic tensor product $X \odot Y$ in $B(H \otimes_2 K)$. It is common to denote $M_n \otimes X$ by $M_n(X)$. Here, $M_n = B(\ell_2^n)$ is the space of $n \times n$ matrices. We view $M_n(X)$ as the space of X -valued $n \times n$ matrices, with the norm $\|\cdot\|_n$. It is easy to see that the sequence of *matricial norms* $\|\cdot\|_n$ satisfies two properties (*Ruan’s axioms*): (i) for any $v \in M_n(X)$, $\alpha \in M_{n,k}$, and $\beta \in M_{k,n}$, $\|(\beta \otimes I_X)v(\alpha \otimes I_X)\|_k \leq \|\beta\|_k \|v\|_n \|\alpha\|_n$, and (ii) for any $v \in M_n(X)$ and $w \in M_k(X)$, $\|v \oplus w\|_{k+n} = \max\{\|v\|_n, \|w\|_k\}$. It turns out that the converse is also true. Suppose X is a *matricially normed space* – that is, it is a Banach space, for which the spaces $M_n(X)$ of $n \times n$ X -valued matrices are equipped with the norms $\|\cdot\|_n$, satisfying (i) and (ii) above. Then the norms $\|\cdot\|_n$ arise from an isometric embedding of X into $B(H)$, for some Hilbert space H . The spaces X as above are sometimes called *abstract operator spaces*.

It is easy to see that a subspace of an operator space is, again, an operator space (we use the notation \hookrightarrow to denote one operator space being a subspace of another). Moreover,

a quotient and the dual of an operator space can be again equipped with an operator space structure. Once again, the reader is referred to [3, 19, 22] for details.

A map u from an operator space X to an operator space Y is called *completely bounded* (*c.b.* for short) if its *c.b. norm*

$$\|u\|_{cb} = \sup_n \|I_{M_n} \otimes u\|_{B(M_n(X), M_n(Y))} = \|I_{B(\ell_2)} \otimes u\|_{B(B(\ell_2) \otimes X, B(\ell_2) \otimes Y)}$$

is finite. The set of all c.b. maps from X to Y is denoted by $CB(X, Y)$. Clearly, $\|u\|_{cb} \geq \|u\|$, and $CB(X, Y) \subset B(X, Y)$ (the inclusion may be strict). The operator spaces X and Y are called *completely isomorphic* (*completely isometric*) if there exists $u \in CB(X, Y)$ such that u^{-1} is c.b. (resp. $\|u\|_{cb} = \|u^{-1}\|_{cb} = 1$). We shall use the notation \simeq for complete isomorphism. Moreover, we say that X is *c-completely isomorphic to Y* ($X \stackrel{c}{\simeq} Y$) if there exists $u \in CB(X, Y)$ with $\|u\|_{cb}\|u^{-1}\|_{cb} \leq c$. Complete embeddability and biembeddability are defined in the obvious way.

Suppose the operator spaces X and Y are embedded into $B(H)$ and $B(K)$, respectively. We define the *direct sum* of operator spaces X and Y (denoted by $X \oplus_\infty Y$, or simply $X \oplus Y$) by viewing $X \oplus Y$ as embedded into $B(H \oplus_2 K)$. Note that any $u \in M_n(X \oplus Y)$ has a unique expansion as $v \oplus w$, with $v \in M_n(X)$ and $w \in M_n(Y)$. Then $\|u\| = \max\{\|v\|, \|w\|\}$.

Throughout this paper, we work with the row and column spaces. Recall that a Hilbert space H can be equipped with *row* and *column* operator space structure, denoted by $H_{\mathbf{R}}$ and $H_{\mathbf{C}}$, respectively. The space $H_{\mathbf{R}}$ is defined as the linear space of operators $\xi \otimes \xi_0$, where ξ_0 is a fixed unit vector, and ξ runs over H . Here, for $\xi \in H$ and $\eta \in K$, $\xi \otimes \eta$ denotes the operator in $B(H, K)$, defined by $(\xi \otimes \eta)\zeta = \langle \zeta, \xi \rangle \eta$. Similarly, the space $H_{\mathbf{C}}$ is defined as the space of operators $\xi_0 \otimes \xi$ ($\xi \in H$).

It is easy to see that, if K is a subspace of H , then $K_{\mathbf{C}}$ ($K_{\mathbf{R}}$) is a subspace of $H_{\mathbf{C}}$ (resp. $H_{\mathbf{R}}$). For simplicity of notation, we write denoted by \mathbf{R} and \mathbf{C} , instead of $(\ell_2)_{\mathbf{R}}$ and $(\ell_2)_{\mathbf{C}}$, respectively. One can use matrix units to describe these spaces. We denote by $E_{ij} \in B(\ell_2)$ the infinite matrix with 1 on the intersection of the i -th row and the j -th column, and zeroes elsewhere. Then \mathbf{R} (\mathbf{C}) is the closed linear span of $(E_{1j})_{j=1}^\infty$ (respectively, $(E_{i1})_{i=1}^\infty$). Below we list a few useful properties of row and column spaces. Here, H and K are Hilbert spaces.

- (1) $H_{\mathbf{R}}$ and $H_{\mathbf{C}}$ are isometric to H (as Banach spaces).
- (2) For any $u \in B(H, K)$, $\|u\| = \|u\|_{CB(H_{\mathbf{R}}, K_{\mathbf{R}})} = \|u\|_{CB(H_{\mathbf{C}}, K_{\mathbf{C}})}$.

- (3) If (ξ_i) is an orthonormal system in H , then, for any finite sequence (a_i) of elements of M_n ,

$$\begin{aligned} \|\sum_i a_i \otimes \xi_i\|_{M_n(H_{\mathbf{R}})} &= \|(\sum_i a_i a_i^*)^{1/2}\| = \|\sum_i a_i a_i^*\|^{1/2}, \\ \|\sum_i a_j \otimes \xi_i\|_{M_n(H_{\mathbf{C}})} &= \|(\sum_i a_i^* a_i)^{1/2}\| = \|\sum_i a_i^* a_i\|^{1/2}. \end{aligned}$$

- (4) Suppose $H^{(1)}, \dots, H^{(n)}$ are Hilbert spaces. Then the formal identity map $id : (H^{(1)} \oplus_2 \dots \oplus_2 H^{(n)})_{\mathbf{R}} \rightarrow H_{\mathbf{R}}^{(1)} \oplus \dots \oplus H_{\mathbf{R}}^{(n)}$ is a complete contraction, and $\|id^{-1}\|_{cb} \leq \sqrt{n}$. The same is also true for column spaces.
- (5) For any $u \in CB(H_{\mathbf{R}}, K_{\mathbf{C}})$ or $u \in CB(H_{\mathbf{C}}, K_{\mathbf{R}})$, $\|u\|_2 = \|u\|_{cb}$ (here, $\|\cdot\|_2$ is the Hilbert-Schmidt norm).
- (6) Duality: $H_{\mathbf{R}}^* = H_{\mathbf{C}}$, and $H_{\mathbf{C}}^* = H_{\mathbf{R}}$.
- (7) For any operator space X and any $u \in B(X, H_{\mathbf{C}})$ ($u \in B(X, H_{\mathbf{R}})$), $\|u\|_{cb} = \|I_{\mathbf{C}} \otimes u\|_{B(\mathbf{C} \otimes X, \mathbf{C} \otimes H_{\mathbf{C}})}$ (resp. $\|u\|_{cb} = \|I_{\mathbf{R}} \otimes u\|_{B(\mathbf{R} \otimes X, \mathbf{R} \otimes H_{\mathbf{R}})}$). This result follows from the proof of Smith's lemma – see e.g. Proposition 2.2.2 of [3].
- (8) If H is separable infinite dimensional, then $H_{\mathbf{R}}$ ($H_{\mathbf{C}}$) is completely isometric to \mathbf{R} (resp. \mathbf{C}).

A sequence $(x_i)_{i \in I}$ in an operator space X is called *normalized* if $\|x_i\| = 1$ for every $i \in I$. (x_i) is said to be a *c-completely unconditional basic sequence* ($c \geq 1$) if, for any finite sequence of matrices (a_i) , and any sequence of scalars $\lambda_i \in \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$, we have $\|\sum_i \lambda_i a_i \otimes x_i\|_{M_n(X)} \leq c \|\sum_i a_i \otimes x_i\|_{M_n(X)}$. It is easy to see that any such sequence (x_i) is linearly independent. A *c-completely unconditional basic sequence* $(x_i) \subset X$ is called a *c-completely unconditional basis* in X if $X = \text{span}[x_i : i \in I]$ (here and below, $\text{span}[\mathcal{F}]$ refers to the closed linear span of the family \mathcal{F}).

A convenient example of a normalized 1-completely unconditional basis is provided by an orthonormal basis in \mathbf{R} or \mathbf{C} . For future reference, we observe that any completely unconditional basis is “similar to” an orthogonal one. More precisely, suppose X is an operator space, which is isometric to a Hilbert space (this is the setting we are concerned with in this paper). Suppose, furthermore, that (x_i) is a normalized *c-completely unconditional basis* in an operator space X , isometric to a Hilbert space, then (x_i) is “similar to” an orthonormal basis: for any finite sequence of scalars (α_i) ,

$$c^2 \|\sum_i \alpha_i x_i\|^2 \geq \text{Ave}_{\pm} \|\sum_i \pm \alpha_i x_i\|^2 = \sum_i |\alpha_i|^2,$$

and similarly, $c^{-2} \|\sum_i \alpha_i x_i\|^2 \leq \sum_i |\alpha_i|^2$. Thus, there exists an $U : X \rightarrow \ell_2(I)$, s.t. $(Ux_i)_{i \in I}$ is an orthonormal basis, and $\|U\|, \|U^{-1}\| \leq c$.

Families $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ in operator spaces X and Y , respectively, are called *c-equivalent* if there exists a map $u : \text{span}[x_i : i \in I] \rightarrow \text{span}[y_i : i \in I]$, such

that $ux_i = y_i$, and $\|u\|_{cb}\|u^{-1}\|_{cb} \leq c$. Two sequences are *equivalent* if they are c -equivalent, for some c . Furthermore, $(x_i)_{i \in I}$ is c -equivalent to a subfamily of $(y_j)_{j \in J}$ if there exists a subset $I' \subset J$, such that $|I| = |I'|$, and $(x_i)_{i \in I}$ is c -equivalent to $(y_j)_{j \in I'}$. Finally, $(x_i)_{i \in I}$ is c -equivalent to $(y_i)_{i \in I}$ up to a permutation if there exists a bijection $\pi : I \rightarrow I$ such that $(x_i)_{i \in I}$ is c -equivalent to $(y_{\pi(i)})_{i \in I}$.

A sequence $(x_i)_{i \in I}$ is called *completely unconditional* if it is c -completely unconditional, for some c . A completely unconditional basis, equivalence of sequences etc. are defined by dropping the c , in a similar manner.

3 Subspaces of $\mathbf{R} \oplus \mathbf{C}$: basic facts

Suppose H and K are separable Hilbert spaces, and $A \in B(H, K)$ is a contraction. Denote by $X_{\mathbf{R}}(H, K, A)$ the subspace of $H_{\mathbf{R}} \oplus K_{\mathbf{C}}$, spanned by (e, Ae) ($e \in H$). If there is no confusion as to the spaces H and K , we simply write $X_{\mathbf{R}}(A)$. $X_{\mathbf{C}}(A)$ is defined in the same way. Often, we write $X(A)$ ($X(H, K, A)$) instead of $X_{\mathbf{R}}(A)$ ($X_{\mathbf{R}}(H, K, A)$). Note that the formal identity map $id : H \rightarrow X(H, K, A) : \xi \mapsto \xi \oplus A\xi$ is an isometry. Thus, we identify subspaces of $X(H, K, A)$ with those of H . This identification gives meaning to the notation $A|_Y$, where $Y \hookrightarrow X(A)$.

Remark 3.1 Although the spaces \mathbf{R} and \mathbf{C} are “simple”, the structure of their direct sum $\mathbf{R} \oplus \mathbf{C}$ is rather rich. For instance, it was shown in [27] that the “operator Hilbert space” OH is a subspace of a quotient of $\mathbf{R} \oplus \mathbf{C}$ (actually, the results of that paper are much more general). It follows from [27] that the spaces $X(A)$ ($A \in B(\ell_2)$) defined as above are natural “building blocks” of subspaces of $\mathbf{R} \oplus \mathbf{C}$.

Begin by stating a simple lemma.

Lemma 3.2 Suppose H, K , and K' are Hilbert spaces, and $A \in B(H, K)$ is a contraction.

- (1) Suppose $U \in B(K, K')$ is such that $\|U\xi\| = \|\xi\|$ for any $\xi \in \text{ran } A$. Then $X(H, K, A)$ is completely isometric to $X(H, K', UA)$. In particular, $X(H, K, A)$ is completely isometric to $X(H, H, |A|)$, where $|A| = (A^*A)^{1/2}$.
- (2) Suppose P_1, \dots, P_m and Q_1, \dots, Q_m are families of orthogonal mutually orthogonal projections on H and K , respectively, such that $\sum_k P_k = I_H$, $\sum_k Q_k = I_K$, and $A = \sum_{k=1}^m Q_k A P_k$. Set $H_k = P_k(H)$ and $K_k = Q_k(K)$. Then the formal identity operator id from $X(H, K, A)$ to $X(H_1, K_1, Q_1 A P_1) \oplus \dots \oplus X(H_m, K_m, Q_m A P_m)$ is a complete contraction, and $\|id^{-1}\|_{cb} \leq \sqrt{m}$.

(3) $X(A)$ is $|\lambda|^{-1}$ -completely isomorphic to $X(\lambda A)$ whenever $0 < |\lambda| \leq 1$.

Proof We only establish part (2). A Gram-Schmidt orthogonalization shows that any element $x \in M_n(H)$ can be written as $x = x_1 + \dots + x_n$, with $x_k = \sum_i a_{ki} \otimes \xi_{ki}$ $a_{ki} \in M_n$, and (ξ_{ki}) a finite orthonormal systems in H_k . Then

$$\|x\|_{M_n(X(A))} = \max \left\{ \left\| \sum_k \sum_i a_{ki} \otimes \xi_{ki} \right\|_{M_n(H_{\mathbf{R}})}, \left\| \sum_k \sum_i a_{ki} \otimes A\xi_{ki} \right\|_{M_n(K_{\mathbf{C}})} \right\}.$$

By the basic properties of row and column spaces (listed at the end of the previous section),

$$\begin{aligned} \max_k \left\| \sum_i a_{ki} \otimes \xi_{ki} \right\|_{M_n(H_{\mathbf{R}})} &\leq \left\| \sum_{k,i} a_{ki} \otimes \xi_{ki} \right\|_{M_n(H_{\mathbf{R}})} \\ &\leq \sqrt{m} \max_k \left\| \sum_i a_{ki} \otimes \xi_{ki} \right\|_{M_n(H_{\mathbf{R}})}. \end{aligned}$$

Furthermore, the vectors $A\xi_{ki}$ belong to the mutually orthogonal spaces K_k , hence

$$\begin{aligned} \max_k \left\| \sum_i a_{ki} \otimes A\xi_{ki} \right\|_{M_n(H_{\mathbf{C}})} &\leq \left\| \sum_{k,i} a_{ki} \otimes A\xi_{ki} \right\|_{M_n(H_{\mathbf{C}})} \\ &\leq \sqrt{m} \max_k \left\| \sum_i a_{ki} \otimes A\xi_{ki} \right\|_{M_n(H_{\mathbf{C}})}. \end{aligned}$$

Note that

$$\begin{aligned} &\|x_k\|_{M_n(X(H_k, Q_k, Q_k A P_k))} \\ &= \max \left\{ \left\| \sum_i a_{ki} \otimes \xi_{ki} \right\|_{M_n(H_{\mathbf{R}})}, \left\| \sum_i a_{ki} \otimes A\xi_{ki} \right\|_{M_n(H_{\mathbf{C}})} \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\max_k \|x_k\|_{M_n(X(H_k, Q_k, Q_k A P_k))} \\ &= \max \left\{ \max_k \left\| \sum_i a_{ki} \otimes \xi_{ki} \right\|_{M_n(H_{\mathbf{R}})}, \max_k \left\| \sum_i a_{ki} \otimes A\xi_{ki} \right\|_{M_n(H_{\mathbf{C}})} \right\} \\ &\leq \|x\|_{M_n(X(A))} \\ &\leq \sqrt{m} \max \left\{ \max_k \left\| \sum_i a_{ki} \otimes \xi_{ki} \right\|_{M_n(H_{\mathbf{R}})}, \max_k \left\| \sum_i a_{ki} \otimes A\xi_{ki} \right\|_{M_n(H_{\mathbf{C}})} \right\}. \end{aligned}$$

We complete the proof by noting that

$$\|x\|_{M_n(X(H_1, K_1, Q_1 A P_1) \oplus \dots \oplus X(H_m, K_m, Q_m A P_m))} = \max_{1 \leq k \leq m} \|x_k\|_{M_n(X(H_k, Q_k, Q_k A P_k))}. \quad \square$$

Next, we establish our key tool for computing c.b. norms.

Lemma 3.3 Suppose A and B are contractions, and $T \in B(X(A), X(B))$. Then

$$\|T\|_{cb} = \max \{ \|T\|, \sup \{ \|BTu\|_2 : u \in B(\ell_2, X(A)), \|Au\|_2 \leq 1, \|u\| \leq 1 \} \}.$$

Proof By definition, $\|T\|_{cb} = \max \{ \|T\|_{CB(X(A), \mathbf{R})}, \|BT\|_{CB(X(A), \mathbf{C})} \}$. To estimate the first term, note that $id : X(A) \rightarrow \mathbf{R}$ is a complete contraction, hence

$$\|T\|_{CB(X(A), \mathbf{R})} = \|T \circ id\|_{CB(X(A), \mathbf{R})} \leq \|T\|_{CB(\mathbf{R}, \mathbf{R})} \|id\|_{CB(X(A), \mathbf{R})} = \|T\|.$$

However, $\|T\| \leq \|T\|_{CB(X(A), \mathbf{R})}$, hence $\|T\| = \|T\|_{CB(X(A), \mathbf{R})}$.

Next we estimate $\|BT\|_{CB(X(A), \mathbf{C})}$. We know that

$$\|BT\|_{CB(X(A), \mathbf{C})} = \sup \{ \|(I_{\mathbf{C}} \otimes BT)(x)\|_{\mathbf{C} \otimes \mathbf{C}} : x \in \mathbf{C} \otimes X(A), \|x\|_{\mathbf{C} \otimes X(A)} \leq 1 \}.$$

Identifying elements of $\mathbf{C} \otimes X(A)$ and $\mathbf{C} \otimes \mathbf{C}$ with operators from \mathbf{R} to $X(A)$ and \mathbf{C} , respectively, we see that

$$\|BT\|_{CB(X(A), \mathbf{C})} = \sup \{ \|BTu\|_2 : u \in CB(\mathbf{R}, X(A)), \|u\|_{cb} \leq 1 \}.$$

But $\|u\|_{cb} = \max \{ \|u\|_{CB(\mathbf{R})}, \|Au\|_{CB(\mathbf{R}, \mathbf{C})} \} = \max \{ \|u\|, \|Au\|_2 \}$. □

Next we examine the exactness of $X(A)^*$. Recall that an operator space X is called *exact* if there exists $c > 0$ such that, for any finite dimensional subspace E of X , there exists an operator u from E to a subspace F of M_n ($n \in \mathbb{N}$), such that $\|u\|_{cb} \|u^{-1}\|_{cb} < c$. The infimum of all the c 's like this is called the *exactness constant* of X . It is easy to see that $\mathbf{R} \oplus \mathbf{C}$ is 1-exact, hence so is $X(A)$. The case of the dual is different.

Proposition 3.4 Suppose $A \in B(H, K)$ is a contraction. Then the exactness constant of $X(A)^*$ is at least $2^{-5/2} \|A\|_2$. In particular, $X(A)^*$ is not exact if A is not Hilbert-Schmidt.

Proof Let $c = 2^{5/2} \text{ex}(X(A)^*)$. By Corollary 0.7 of [23], there exist operators $T_1 : X(A) \rightarrow \mathbf{R}$, $T_2 : \mathbf{R} \rightarrow X(A)$, $S_1 : X(A) \rightarrow \mathbf{C}$, and $T_2 : \mathbf{C} \rightarrow X(A)$, such that $id = S_2 S_1 + T_2 T_1$ (id is the identity on $X(A)$), and

$$\|S_1\|_{cb} \|S_2\|_{cb} + \|T_1\|_{cb} \|T_2\|_{cb} \leq \max \{ \|S_1\|_{cb}, \|T_1\|_{cb} \} (\|S_2\|_{cb} + \|T_2\|_{cb}) \leq c$$

By Lemma 3.3, $\|T_2\|_{cb} \geq \|AT_2\|_2$, and a simple calculation yields

$$\|S_2\|_{cb} \geq \|S_2\|_{CB(\mathbf{C}, \mathbf{R})} = \|S_2\|_2 \geq \|AS_2\|_2.$$

Therefore,

$$\begin{aligned} \|S_1\|_{cb} \|S_2\|_{cb} + \|T_1\|_{cb} \|T_2\|_{cb} &\geq \|S_1\| \|AS_2\|_2 + \|T_1\| \|AT_2\|_2 \\ &\geq \|A(S_2 S_1 + T_2 T_1)\|_2 = \|A\|_2, \end{aligned}$$

which implies the desired estimate for c . □

Corollary 3.5 *The space $X(A)$ is completely isomorphic to \mathbf{R} if and only if A is Hilbert-Schmidt. Furthermore, the formal identity map $id : \mathbf{R} \rightarrow X(A)$ is completely contractive, and $\|id^{-1}\|_{cb} = \max\{1, \|A\|_2\}$.*

Proof The estimates on the c.b. norm of id and id^{-1} follow from Lemma 3.3. Thus, $X(A) \stackrel{\max\{1, \|A\|_2\}}{\simeq} \mathbf{R}$ whenever A is Hilbert-Schmidt. On the other hand, if A is not Hilbert-Schmidt, then, by Proposition 3.4, $X(A)^*$ is not exact. However, $\mathbf{R}^* = \mathbf{C}$ is exact, thus $X(A)$ is not completely isomorphic to \mathbf{R} . \square

We say that $A \in B(H, K)$ (H and K are Hilbert spaces) is *diagonalizable* if the eigenvectors of $|A| = (A^*A)^{1/2}$ span H . Equivalently, there exist orthonormal systems (ξ_i) and (η_i) in H and K , respectively, and a sequence (λ_i) of non-negative numbers, such that $A = \sum_i \lambda_i \xi_i \otimes \eta_i$.

When working with $X(A)$, it is often convenient to have A diagonalizable. While every compact operator is diagonalizable, a non-compact one need not have this property. However, we have:

Lemma 3.6 *Suppose H and K are separable Hilbert spaces, $A \in B(H, K)$ is a contraction, and $\varepsilon > 0$. Then there exists a diagonalizable contraction $B \in B(H, K)$ such that $\|A - B\|_2 \leq \varepsilon$, and $X(A)$ is $(1 + \varepsilon)^2$ -completely isomorphic to $X(B)$. If A is non-negative, B can be selected to be non-negative, too.*

Proof By Lemma 3.2(1), it suffices to consider the case of $A = |A|$, and $H = K$. By [26], there exists a selfadjoint diagonalizable $C \in B(H)$ such that $\|A - C\|_2 < \varepsilon/2$. Let (c_i) be the eigenvalues of C , and (ξ_i) the corresponding norm 1 eigenvectors. Define the operator B by setting $B\xi_i = b_i\xi_i$, where

$$b_i = \begin{cases} c_i & 0 \leq c_i \leq 1 \\ 0 & c_i < 0 \\ 1 & c_i > 1 \end{cases}.$$

We claim that $\|B - C\|_2 < \varepsilon/2$. Indeed, let $\mathcal{I} = \{i : b_i \neq c_i\}$. Then

$$\|B - C\|_2^2 = \sum_{i \in \mathcal{I}} |b_i - c_i|^2 = \sum_{i: c_i < 0} |c_i|^2 + \sum_{i: c_i > 1} |b_i|^2.$$

But $\langle \xi_i, A\xi_i \rangle \in [0, 1]$, hence, for $c_i < 0$, $|\langle \xi_i, A\xi_i \rangle - \langle \xi_i, C\xi_i \rangle| \geq |c_i|$. Similarly, for $c_i > 1$, $|\langle \xi_i, A\xi_i \rangle - \langle \xi_i, C\xi_i \rangle| \geq |1 - c_i|$. Thus,

$$\frac{\varepsilon^2}{4} > \|A - C\|_2^2 = \sum_{i,j} |\langle \xi_i, (A - C)\xi_j \rangle|^2 \geq \sum_i |\langle \xi_i, (A - C)\xi_i \rangle|^2 \geq \sum_{i \in \mathcal{I}} |b_i - c_i|^2.$$

By the triangle inequality, $\|A - B\|_2 < \varepsilon$. Moreover, B is a non-negative contraction. Denote the the formal identity map from $X(A)$ to $X(B)$ by U . It remains to show that $\|U\|_{cb}, \|U^{-1}\|_{cb} \leq 1 + \varepsilon$. As U is an isometry, Lemma 3.3 implies

$$(3-1) \quad \|U\|_{cb} = \sup\{\|Bu\|_2 : u \in B(\ell_2, X(A)), \|Au\|_2 \leq 1, \|u\| \leq 1\}.$$

By the triangle inequality,

$$\|Bu\|_2 \leq \|Au\|_2 + \|(B - A)u\|_2 \leq \|Au\|_2 + \|B - A\|_2 \leq 1 + \varepsilon,$$

hence (3-1) yields $\|U\|_{cb} \leq 1 + \varepsilon$. $\|U^{-1}\|_{cb}$ is estimated similarly. \square

Proposition 3.7 *Suppose H and K are separable Hilbert spaces, $B \in B(H, K)$ is a contraction, and $0 \in \sigma_{ess}(|B|)$. Then $X(B)$ is $4\sqrt{2}$ -completely isomorphic to $X(B) \oplus \mathbf{R}$.*

Proof We can assume $B = |B|$. By Lemma 3.6, there exists a non-negative diagonalizable contraction A such that $A - B$ is Hilbert-Schmidt, and $X(A) \stackrel{2^{1/4}}{\simeq} X(B)$. As the essential spectrum is stable under compact perturbations, $0 \in \sigma_{ess}(A)$. Write $A = \text{diag}(\alpha)$, where $\alpha = (\alpha_i)_{i \in I}$, and $\text{diag}(\alpha) \in B(\ell_2)$ is the diagonal operator with α, α_2, \dots on the main diagonal. Then 0 is a cluster point of the set (α_i) . Denote the norm 1 eigenvectors, corresponding to α_i , by ξ_i . Find $I_0 \subset I$ such that $\sum_{i \in I_0} \alpha_i^2 < 1$. Let $I_1 = I \setminus I_0$, and let P_0 and P_1 be the orthogonal projections onto $H_0 = \text{span}[\xi_i : i \in I_0]$ and $H_1 = \text{span}[\xi_i : i \in I_1]$, respectively. By Lemma 3.2, $X(A)$ is $\sqrt{2}$ -completely isomorphic to $X(H_0, H_0, P_0AP_0) \oplus X(H_1, H_1, P_1AP_1)$. By Corollary 3.5, $X(H_0, H_0, P_0AP_0)$ is completely isometric to \mathbf{R} . Thus,

$$\begin{aligned} X(A) &\stackrel{\sqrt{2}}{\simeq} X(H_0, H_0, P_0AP_0) \oplus X(H_1, H_1, P_1AP_1) \\ &\stackrel{\sqrt{2}}{\simeq} \mathbf{R} \oplus X(H_1, H_1, P_1AP_1) \stackrel{\sqrt{2}}{\simeq} \mathbf{R} \oplus \mathbf{R} \oplus X(H_1, H_1, P_1AP_1) \\ &= \mathbf{R} \oplus X(H_0, H_0, P_0AP_0) \oplus X(H_1, H_1, P_1AP_1) \stackrel{\sqrt{2}}{\simeq} \mathbf{R} \oplus X(A). \end{aligned}$$

To summarize, $X(A) \stackrel{4}{\simeq} X(A) \oplus \mathbf{R}$. As $X(A) \stackrel{2^{1/4}}{\simeq} X(B)$, we are done. \square

Corollary 3.8 *Suppose H and K are separable infinite dimensional Hilbert spaces, and a contraction $A \in B(H, K)$ satisfies $0 \in \sigma_{ess}(|A|)$. Consider the operator $\tilde{A} = A \oplus 0$ from $\tilde{H} = H \oplus_2 \ell_2$ to $\tilde{K} = K \oplus_2 \ell_2$. Then $X(A) \stackrel{8}{\simeq} X(\tilde{A})$.*

Proof Let P be the orthogonal projection from \tilde{H} onto H . By Lemma 3.2(2) and Corollary 3.5, $X(\tilde{A}) \stackrel{\sqrt{2}}{\simeq} X(A) \oplus \mathbf{R}$. However, by Proposition 3.7, $X(A) \stackrel{4\sqrt{2}}{\simeq} X(A) \oplus \mathbf{R}$. \square

4 Classification of subspaces using sequences

In this section, we study the spaces $X(A)$ when A is compact, and establish a connection between such spaces and a certain family of sequences. We denote by \mathfrak{C} the space of compact contractions $A \in B(\ell_2)$, which are not Hilbert-Schmidt. We denote by \mathfrak{X} the set of all spaces $X(A)$, with $A \in \mathfrak{C}$.

Start by defining the canonical basis in $X(A)$, where $A \in B(H, K)$ is a compact contraction. As $X(A) = X(|A|)$, we assume henceforth that $A = |A|$, and $H = K$. Let $(\xi_i)_{i \in I_1}$ be the normalized eigenvectors of A , corresponding to the positive eigenvalues of A . Furthermore, set $H' = \ker A$, and let $(\xi_i)_{i \in I_0}$ be an orthonormal basis in H' (we assume that $I_1 \cap I_0 = \emptyset$). Let $I = I_0 \cup I_1$. The vectors $e_i = \xi_i \oplus A\xi_i \in H_{\mathbf{R}} \oplus H_{\mathbf{C}}$ ($i \in I$) span $X(A)$. Moreover, $\|e_i\| = 1$ for each i , and, for any finite sequence $(a_i) \subset M_n$,

$$(4-1) \quad \left\| \sum_i a_i \otimes e_i \right\|_{M_n(X(A))}^2 = \max \left\{ \left\| \sum_i a_i a_i^* \right\|, \left\| \sum_i \|A\xi_i\|^2 a_i^* a_i \right\| \right\}.$$

We say that the vectors $e_i = e_i[A]$ form the *canonical basis* of $X(A)$.

Now suppose $(\alpha_i)_{i \in \mathbb{N}}$ is a sequence of numbers in $[0, 1]$. In an effort to link operator spaces with certain sequences of scalars, we define the operator space $X^d(\alpha) = X(\text{diag}(\alpha))$. To describe the operator space structure of $X^d(\alpha)$, denote by (ξ_i) the canonical orthonormal basis of ℓ_2 . Then the vectors $e_i(\alpha) = e_i = \xi_i \oplus \alpha_i \xi_i \in H_{\mathbf{R}} \oplus K_{\mathbf{C}}$ form a 1-completely unconditional orthonormal basis in $X^d(\alpha)$, with

$$(4-2) \quad \left\| \sum_i a_i \otimes e_i \right\|^2 = \max \left\{ \left\| \sum_i a_i a_i^* \right\|, \left\| \sum_i \alpha_i^2 a_i^* a_i \right\| \right\}$$

for any finite sequence of matrices (a_i) . We call $(e_i(\alpha))_{i \in \mathbb{N}}$ the *canonical basis* of $X^d(\alpha)$. The *formal identity* from $X^d(\alpha)$ to $X^d(\beta)$ is the linear operator mapping $e_i(\alpha)$ to $e_i(\beta)$.

If $\alpha = (\alpha_i)_{i=1}^n$ is a finite sequence, we define the finite dimensional space $X^d(\alpha)$ in the same way.

To reduce ourselves to working with the spaces $X^d(\alpha)$ (and hence to sequences of scalars), define, for a compact $A \in B(H, K)$, the sequence $\alpha = \mathbf{D}(A)$: if A has rank $n < \infty$, let $\alpha_1 \geq \dots \geq \alpha_n > 0$ be the non-zero singular values of A , and set $\alpha_i = 0$ for $i > n$. If the rank of A is infinite, let (α_i) be the singular values of A , listed in the non-increasing order. We have:

Proposition 4.1 *If H and K are separable infinite dimensional Hilbert spaces, and $A \in B(H, K)$ is a compact contraction, then $X(A)$ is 2^6 -completely isomorphic to $X^d(\mathbf{D}(A))$.*

Proof Let $\alpha = (\alpha_i) = \mathbf{D}(A)$. By Lemma 3.2(1), we can assume that $A = |A| = (A^*A)^{1/2}$, and $H = K$. If A has finite rank, (4-1) shows that $X(A)$ is completely isometric to $X^d(\alpha)$. Otherwise, write $A = \sum_i \alpha_i \xi_i \otimes \xi_i$, for some orthonormal system $(\xi_i)_{i=1}^\infty$ in H . Let P be the orthogonal projection onto $K = \text{span}[\xi_i : i \in \mathbb{N}]$. Set $Q = I - P$, and $L = Q(H)$. By Lemma 3.2(2), $X(A)$ is $\sqrt{2}$ -completely isomorphic to $X(K, K, PAP) \oplus X(L, L, QAQ)$. Furthermore, $QAQ = 0$, hence $X(L, L, QAQ) = L_{\mathbf{R}}$. It is easy to see that $X(K, K, PAP) = X^d(\alpha)$. By Proposition 3.7, the latter space is $4\sqrt{2}$ -completely isomorphic to $X^d(\alpha) \oplus \mathbf{R}$. Thus,

$$\begin{aligned} X(A) &\overset{\sqrt{2}}{\simeq} X(K, K, PAP) \oplus X(L, L, QAQ) \overset{4\sqrt{2}}{\simeq} X^d(\alpha) \oplus \mathbf{R} \oplus L_{\mathbf{R}} \\ &\overset{\sqrt{2}}{\simeq} X^d(\alpha) \oplus (\ell_2 \oplus L)_{\mathbf{R}} = X^d(\alpha) \oplus \mathbf{R} \overset{4\sqrt{2}}{\simeq} X^d(\alpha). \end{aligned} \quad \square$$

We will also use a related observation.

Lemma 4.2 For every compact contraction $A \in B(\ell_2)$, and every $\varepsilon > 0$, there exists $\alpha \in c_0$ such that $X(A) \overset{1+\varepsilon}{\simeq} X^d(\alpha)$.

Proof Indeed, let $\beta = \mathbf{D}(A)$. If A is finite rank, then $X(A)$ is completely isometric to $X^d(\beta)$. If $\text{rank } A = \infty$, assume (by Lemma 3.2) that $A = |A|$. Then $\beta_1 \geq \beta_2 \geq \dots > 0$ is the list of all positive eigenvalues of A . Denote the corresponding norm 1 eigenvectors by ξ_i . Let $H = \text{span}[\xi_i : i \in \mathbb{N}]$, and $K = \ker A$. Clearly, K and H are mutually orthogonal subspaces of ℓ_2 . Let $(\eta_j)_{j \in J}$ be the orthogonal basis of K . Find a sequence $(\gamma_j)_{j \in J}$ of positive numbers, satisfying $\sum_{j \in J} \gamma_j^2 < \varepsilon^2$. Consider the compact contraction $\tilde{A} \in B(\ell_2)$, defined by $\tilde{A}\xi_i = \beta_i \xi_i$, and $\tilde{A}\eta_j = \gamma_j \eta_j$. Then $\|A - \tilde{A}\|_2 < \varepsilon$. By (4-1), the formal identity map $id : X(\tilde{A}) \rightarrow X(A)$ is a complete contraction, and $\|id^{-1}\|_{cb} < 1 + \varepsilon$. We complete the proof by identifying $X(\tilde{A})$ with the space $X^d(\alpha)$, where the sequence $\alpha = (\alpha_i)$ is the “join” of the sequences β and γ (that is, any number $c \in [0, 1]$ occurs in α as many times as it occurs in the sequences β and γ combined). \square

Now denote by \mathcal{S} the set of all sequences $(\alpha_i)_{i \in \mathbb{N}}$ satisfying $1 \geq \alpha_1 \geq \alpha_2 \geq \dots \geq 0$, and $\lim_i \alpha_i = 0$. The rest of this section is devoted to the spaces $X^d(\alpha)$ ($\alpha \in \mathcal{S}$). We translate the relations between sequences $\alpha, \beta \in \mathcal{S}$ to relations between the corresponding spaces $X^d(\alpha)$ and $X^d(\beta)$. We say that the sequence α dominates β ($\alpha \succ \beta$) if there exists a set $S \subset \mathbb{N}$ and $K > 0$ s.t. $\sum_{i \in S} \beta_i^2 < \infty$, and $K\alpha_i \geq \beta_i$ for any $i \notin S$. We say α is equivalent to β ($\alpha \sim \beta$) if $\alpha \succ \beta$, and $\beta \succ \alpha$.

Clearly, the relation \succ is reflexive and transitive. The relation \sim is, in addition to this, symmetric. For instance, to establish the transitivity of \succ , suppose $\alpha \succ \beta$, and $\beta \succ \gamma$,

and show that $\alpha \succ \gamma$. Note that there exist sets S_1 and S_2 , and constants K_1 and K_2 , s.t. $\beta_i \leq K_1 \alpha_i$ for $i \notin S_1$, $\gamma_i \leq K_2 \beta_i$ for $i \notin S_2$, $\sum_{i \in S_1} \beta_i^2 < \infty$, and $\sum_{i \in S_2} \gamma_i^2 < \infty$. Let $S = S_1 \cup S_2$, and $K = K_1 K_2$. Then $\gamma_i \leq K \alpha_i$ for $i \notin S$. Moreover,

$$\sum_{i \in S} \gamma_i^2 = \sum_{i \in S_2} \gamma_i^2 + \sum_{i \in S_1 \setminus S_2} \gamma_i^2 \leq \sum_{i \in S_2} \gamma_i^2 + \sum_{i \in S_1} K_2^2 \beta_i^2 < \infty,$$

which is what we need. The other properties are proved in a similar fashion.

Proposition 4.3 For $\alpha, \beta \in \mathcal{S}$, $\alpha \sim \beta$ if and only if there exists a set $S \subset \mathbb{N}$ and a constant K s.t. $\sum_{i \in S} (\alpha_i^2 + \beta_i^2) < \infty$, and $K^{-1} \alpha_i \leq \beta_i \leq K \alpha_i$ for $i \notin S$.

Proof If S and K with the properties described above exist, then they witness the fact that $\alpha \prec \beta$ and $\alpha \succ \beta$, hence $\alpha \sim \beta$. Conversely, suppose $\alpha \prec \beta$ and $\alpha \succ \beta$. Then there exist a constant K , and sets S_1 and S_2 , s.t. $\alpha_i \leq K \beta_i$ for $i \notin S_1$, $\beta_i \leq K \alpha_i$ for $i \notin S_2$, $\sum_{i \in S_1} \alpha_i^2 < \infty$, and $\sum_{i \in S_2} \beta_i^2 < \infty$. By reducing S_1 and S_2 further, we can assume that $\alpha_i > K \beta_i$ for each $i \in S_1$, and $\beta_i > K \alpha_i$ for each $i \in S_2$. Then $\sum_{i \in S_1} \beta_i^2 < \infty$, and $\sum_{i \in S_2} \alpha_i^2 < \infty$. Therefore, $S = S_1 \cup S_2$ has the required properties. \square

The main result of this section is:

Theorem 4.4 For $\alpha, \beta \in \mathcal{S}$, $X^d(\alpha)$ embeds completely isomorphically into $X^d(\beta)$ if and only if $\alpha \prec \beta$.

From this we immediately obtain

Corollary 4.5 Suppose $\alpha, \beta \in \mathcal{S}$. The following three statements are equivalent.

- (1) $X^d(\alpha)$ is completely isomorphic to $X^d(\beta)$.
- (2) $X^d(\alpha)$ embeds completely isomorphically into $X^d(\beta)$, and vice versa.
- (3) $\alpha \sim \beta$.

Proof (1) \Rightarrow (2) is trivial, while (2) \Rightarrow (3) follows from Theorem 4.4 and Proposition 4.3. To establish (3) \Rightarrow (1), suppose $\alpha \sim \beta$. By Proposition 4.3, there exists a set \mathcal{I} and $K > 0$ s.t. $\sum_{i \in \mathcal{I}} (\alpha_i^2 + \beta_i^2) < \infty$, and $K^{-1} \alpha_i \leq \beta_i \leq K \alpha_i$ for any $i \notin \mathcal{I}$. By Corollary 3.5, the spaces $E_\alpha = \text{span}[e_i : i \in \mathcal{I}] \hookrightarrow X^d(\alpha)$ and $E_\beta = \text{span}[e_i : i \in \mathcal{I}] \hookrightarrow X^d(\beta)$ are completely isomorphic to \mathbf{R} . By (4–2), the formal identity map from $F_\alpha = \text{span}[e_i : i \notin \mathcal{I}] \hookrightarrow X^d(\alpha)$ to $F_\beta = \text{span}[e_i : i \notin \mathcal{I}] \hookrightarrow X^d(\beta)$ is a complete isomorphism. By Lemma 3.2(2), $X^d(\alpha) \simeq E_\alpha \oplus F_\alpha$, and $X^d(\beta) \simeq E_\beta \oplus F_\beta$. Therefore, the formal identity map from $X^d(\alpha)$ to $X^d(\beta)$ is a complete isomorphism. \square

The proof of Theorem 4.4 follows from the next two lemmas.

Lemma 4.6 *If $\alpha, \beta \in \mathcal{S}$ and $\beta \prec \alpha$, then $X^d(\beta)$ embeds completely isomorphically into $X^d(\alpha)$.*

Proof By Corollary 3.8, $X^d(\alpha)$ is completely isomorphic to $X(A)$, where $A \in B(\ell_2 \oplus \ell_2)$ is defined by $A = \text{diag}(\alpha) \oplus 0$. Let (ξ_i) and (ξ'_i) be orthonormal bases in the first and second copies of ℓ_2 , respectively. Then $X(A)$ is the closed linear span of the vectors $\xi_i \oplus \alpha_i \xi_i$ and $\xi'_i \oplus 0$ ($i \in \mathbb{N}$). Find a sequence $\phi_i \in [0, \pi/2]$ s.t. $\beta_i = \cos \phi_i \cdot \alpha_i$. Define an orthonormal system $\eta_i = \cos \phi_i \xi_i + \sin \phi_i \xi'_i$. For $i \in \mathbb{N}$ consider

$$f_i = \eta_i \oplus \beta_i \xi_i = \cos \phi_i (\xi_i \oplus \alpha_i \xi_i) + \sin \phi_i (\xi'_i \oplus 0).$$

Then $f_i \in X(A)$, and $\text{span}\{f_i : i \in \mathbb{N}\} = X^d(\beta)$. □

Lemma 4.7 *Suppose $\alpha, \beta \in \mathcal{S}$, E is a subspace of $X^d(\alpha)$, and a completely bounded map $U : E \rightarrow X^d(\beta)$ has bounded inverse (in the terminology of [18], E is completely semi-isomorphic to $X^d(\beta)$). Then $\beta \prec \alpha$.*

Proof We rely heavily on Wielandt's Minimax Theorem ([2, Theorem III.6.5]): if $c_1 \geq c_2 \geq \dots \geq 0$ are eigenvalues of positive compact operator T , then, for any finite increasing sequence $i_1 < \dots < i_k$ of positive integers,

$$\sum_{j=1}^k c_{i_j} = \sup_{E_1 \hookrightarrow \dots \hookrightarrow E_k} \min_{x_j \in E_j, (x_j) \text{ orthonormal}} \langle Tx_j, x_j \rangle,$$

where the supremum is taken over all subspaces $E_1 \hookrightarrow \dots \hookrightarrow E_k$ of the domain of T , with $\dim E_j = i_j$ for $1 \leq j \leq k$. Actually, the theorem is stated in [2] for operators on finite dimensional spaces, but a generalization to compact operators is easy to obtain. Applying the above identity to $T = S^*S$, where S is a compact operator with singular numbers $s_1 \geq s_2 \geq \dots \geq 0$, we obtain:

$$(4-3) \quad \sum_{j=1}^k s_{i_j}^2 = \sup_{E_1 \hookrightarrow \dots \hookrightarrow E_k, \dim E_j = i_j} \min_{x_j \in E_j, (x_j) \text{ orthonormal}} \|Sx_j\|^2.$$

In our situation, assume $\|U\|_{cb} = 1$. Let $c = \|U^{-1}\|$, $A = \text{diag}(\alpha_i)$, $B = \text{diag}(\beta_i)$, $B' = BU$. Denote the singular values of B' by (β'_i) . Clearly, $\beta'_i \leq \beta_i \leq c\beta'_i$ for every $i \in \mathbb{N}$.

Pick an orthonormal system $(x_j)_{j=1}^k$ in E (the domain of U). Let u be the formal identity from $\mathbf{R}^k = (\ell_2^k)_{\mathbf{R}}$ (the k -dimensional row space) to $\text{span}[x_j : 1 \leq j \leq k]$. Then (compare with the proof of Lemma 3.3)

$$\|u\|_{cb}^2 = \max\{1, \|Au\|_2^2\} \leq 1 + \sum_{j=1}^k \|Ax_j\|^2,$$

and (since U is a complete contraction)

$$\|u\|_{cb}^2 \geq \|Uu\|_{cb}^2 \geq \|BUu\|_2^2 = \sum_{j=1}^k \|B'x_j\|^2.$$

Thus, $\sum_{j=1}^k \|B'x_j\|^2 \leq \sum_{j=1}^k \|Ax_j\|^2 + 1$ for any orthonormal family $(x_j)_{j=1}^k$. By (4-3),

$$(4-4) \quad 1 + \sum_{j=1}^k \alpha_{i_j}^2 \geq \sum_{j=1}^k \beta_{i_j}'^2$$

for any $i_1 < \dots < i_k$ (indeed, when computing $\sum_{j=1}^k \alpha_{i_j}^2$, we are taking the supremum over a larger family of subspaces (E_j) , than when we are computing $\sum_{j=1}^k \beta_{i_j}'^2$).

Now let $\mathcal{I} = \{i \in \mathbb{N} : \beta_i' > 2\alpha_i\}$. Then $\sum_{i \in \mathcal{I}} \beta_i'^2 \leq 2$. Indeed, otherwise there exists a sequence $i_1 < \dots < i_k$ of elements of \mathcal{I} s.t. $C = \sum_{j=1}^k \beta_{i_j}'^2 > 2$. Then, by (4-4), $\sum_{j=1}^k \alpha_{i_j}^2 \geq C - 1$. On the other hand, $\sum_{j=1}^k \alpha_{i_j}^2 \leq C/2$, a contradiction. \square

Developing the ideas of this proof, we obtain:

Theorem 4.8 *Suppose $\alpha, \beta \in \mathcal{S}$, and there exists an isomorphism $U : X^d(\alpha) \rightarrow X^d(\beta)$ with $\|U\|_{cb}, \|U^{-1}\|_{cb} \leq C$. Then the formal identity map $id : X^d(\alpha) \rightarrow X^d(\beta)$ satisfies $\|id\|_{cb}, \|id^{-1}\|_{cb} < 4C^2$.*

Proof As in the proof of Lemma 4.7, let $A = \text{diag}(\alpha_i)$, $B = \text{diag}(\beta_i)$, and $B' = BU$. By Lemma 3.3, $\|B'u\|_2 \leq C \max\{\|Au\|_2, \|u\|\}$ for any $u : \ell_2 \rightarrow X^d(\alpha)$. Denote the singular numbers of B' by (β_i') , and note that $\beta_i/C \leq \beta_i' \leq C\beta_i$ for every i . Reasoning as in the proof of Lemma 4.7, we see that

$$C^2(1 + \sum_{j=1}^k \alpha_{i_j}^2) \geq \sum_{j=1}^k \beta_{i_j}'^2$$

Let $\mathcal{I} = \{i : \beta_i' > 2C\alpha_i\}$. As in the preceding proof, $\sum_{i \in \mathcal{I}} \beta_i'^2 < 2C^2$. Therefore, $\sum_{i \in \mathcal{I}} \beta_i'^2 < 2C^4$, and $\beta_i \leq 2C^2\alpha_i$ for $i \notin \mathcal{I}$.

Next we show that $\|id : X^d(\alpha) \rightarrow X^d(\beta)\|_{cb} < 4C^2$. As before, denote the canonical bases in $X^d(\alpha)$ and $X^d(\beta)$ by $(e_i(\alpha))_{i \in \mathbb{N}}$ and $(e_i(\beta))_{i \in \mathbb{N}}$, respectively. By (4-2), $Y_1 = \text{span}[e_i(\alpha) : i \in \mathcal{I}]$ and $Y_0 = \text{span}[e_i(\alpha) : i \notin \mathcal{I}]$ are completely contractively complemented subspaces of $X^d(\alpha)$. Moreover,

$$\begin{aligned} \|id|_{Y_1}\|_{cb} &\leq \|V : (Y_1)_{\mathbf{R}} \rightarrow X^d(\beta)\|_{cb} = \max\{1, \|BV\|_{cb}\} \\ &\leq \max\{1, (\sum_{i \in \mathcal{I}} \beta_i^2)^{1/2}\} < 2C^2 \end{aligned}$$

(here, V is the formal identity from $(Y_1)_{\mathbf{R}}$ to $\text{span}[e_i(\beta) : i \in \mathcal{I}]$). By Lemma 3.3 and (4-2), $\|id|_{Y_0}\|_{cb} < 2C^2$. Therefore,

$$\|id : X^d(\alpha) \rightarrow X^d(\beta)\|_{cb} \leq \|id|_{Y_1}\|_{cb} + \|id|_{Y_0}\|_{cb} < 4C^2.$$

The norm of $id : X^d(\beta) \rightarrow X^d(\alpha)$ is computed the same way. □

5 Completely unconditional bases

In this section, we further investigate bases in spaces $X(A)$, where $A \in B(H, K)$ is a compact contraction. A subspace E of $X(A)$ is isometric to $X(A|_E)$. As $A|_E$ is a compact contraction, (4-1) implies that E has a 1-completely unconditional basis. The key result of this section is Proposition 5.2, establishing the uniqueness of a completely unconditional basis in E (the existence of such a basis has been established by Proposition 4.1). We also show that the canonical basis (and therefore, every completely unconditional basis) in a completely complemented subspace of $X^d(\alpha)$ is equivalent to a subsequence of the canonical basis of $X^d(\alpha)$. Moreover, there exists $\alpha \in \mathcal{S}$ such that the canonical basis in every complemented subspace of $X^d(\alpha)$ is equivalent to a subsequence of the canonical basis of $X^d(\alpha)$ (Theorem 5.6). In general, the last statement need not be true (Remark 5.5).

First we show that any completely unconditional basic sequence in an $X(A)$ space corresponds to a canonical basis of $X^d(\beta)$, for some β .

Proposition 5.1 *Suppose \mathcal{I} is a finite or countable set, and $(e'_i)_{i \in \mathcal{I}}$ is a C -completely unconditional basic sequence in $X(A)$ ($A \in B(\ell_2)$ is a contraction, not necessarily compact). Let $Y = \text{span}[e'_i : i \in \mathcal{I}]$, and define the sequence $\beta = (\beta_i)$ by setting $\beta_i = \|Ae'_i\|$. Consider the operator $T : Y \rightarrow X^d(\beta) : e'_i \mapsto e_i$, where (e_i) is the canonical basis of $X^d(\beta)$. Then $\|T\|_{cb} \leq C$ and $\|T^{-1}\|_{cb} \leq C^2$.*

Proof As noted in Section 2,

$$(5-1) \quad C^{-2} \left\| \sum \alpha_i e'_i \right\|^2 \leq \sum |\alpha_i|^2 \leq C^2 \left\| \sum \alpha_i e'_i \right\|^2$$

for any finite sequence of scalars (α_i) . Thus, $\|T\|, \|T^{-1}\| \leq C$.

Let $B = \text{diag}(\beta)$ (note that $B = B^*$). By Lemma 3.3, it suffices to show that

$$(5-2) \quad \|BTu\|_2 \leq C \max\{\|Au\|_2, \|u\|\}$$

for any $u : \ell_2 \rightarrow Y = X(A|_Y)$, and

$$(5-3) \quad \|AT^{-1}u\|_2 \leq C^2 \max\{\|Bu\|_2, \|u\|\}$$

for any $u : \ell_2 \rightarrow X^d(\beta)$. By Lemma 3.3, the complete unconditionality of (e'_i) implies:

$$(5-4) \quad \begin{aligned} \|A\Lambda u\|_2 &\leq C \max\{\|Au\|_2, \|u\|\}, \\ \|Au\|_2 &\leq C \max\{\|A\Lambda u\|_2, \|u\|\} \end{aligned}$$

whenever $\Lambda = \text{diag}(\lambda_i)$ (that is, $\Lambda e'_i = \lambda_i e'_i$), with $\lambda_i = \pm 1$ for each i , and $u \in B(\ell_2, Y)$. Note that $\|Au\|_2^2 = \text{tr}(A^*Av)$, where $v = uu^*$. Therefore, (5-4) is equivalent to

$$(5-5) \quad \begin{aligned} \text{tr}(\Lambda^*A^*A\Lambda v) &\leq C^2 \max\{\text{tr}(A^*Av), 1\}, \\ \text{tr}(A^*Av) &\leq C^2 \max\{\text{tr}(\Lambda^*A^*A\Lambda v), 1\} \end{aligned}$$

whenever $v \geq 0$ and $\|v\| = 1$. But $\langle \Lambda^*A^*A\Lambda e'_i, e'_j \rangle = \lambda_i \bar{\lambda}_j \langle Ae'_i, Ae'_j \rangle$. Averaging over $\lambda_i = \pm 1$ for each i , we see that $\text{Ave}_\Lambda \Lambda^*A^*A\Lambda = T^*B^2T$ (since $\langle T^*B^2Te'_i, e'_j \rangle = \delta_{ij} \langle Ae'_i, Ae'_j \rangle$, where δ_{ij} is Kronecker's delta). Therefore, by (5-5),

$$\begin{aligned} \text{tr}(T^*B^2Tv) &= \text{Ave}_\Lambda \text{tr}(\Lambda^*A^*A\Lambda v) \\ &\leq \sup_\Lambda \text{tr}(\Lambda^*A^*A\Lambda v) \leq C^2 \max\{\text{tr}(A^*Av), 1\} \end{aligned}$$

whenever v is a positive contraction. Thus, for any contraction u ,

$$\begin{aligned} \|BTu\|_2^2 &= \text{tr}(T^*B^2Tuu^*) \\ &\leq C^2 \max\{\text{tr}(A^*Auu^*), 1\} = C^2 \max\{\|Au\|_2, \|u\|\}^2, \end{aligned}$$

which proves (5-2).

To establish (5-3), we show that, for every $w : \ell_2 \rightarrow Y$, we have

$$(5-6) \quad \|Aw\|_2 \leq C \max\{\|BTw\|_2, \|w\|\}$$

Indeed, let $w = T^{-1}u$. Then, by (5-6) and (5-1),

$$\begin{aligned} \|AT^{-1}u\|_2 &= \|Aw\|_2 \leq C \max\{\|BTw\|_2, \|w\|\} \\ &\leq C \max\{\|Bu\|_2, C\|u\|\} \leq C^2 \max\{\|Bu\|_2, \|u\|\}. \end{aligned}$$

Moreover, (5–6) is equivalent to the following: for any non-negative, norm one $v \in B(X^d(\alpha))$, we have

$$(5-7) \quad C^2 \max\{\text{tr}(T^*B^2Tv), 1\} \geq \text{tr}(A^*Av).$$

As we have established before, $\text{tr}(T^*B^2Tv) = \text{Ave}_\Lambda \text{tr}(\Lambda^*A^*A\Lambda v)$. By the linearity and positivity of the trace,

$$0 \leq \inf_\Lambda \text{tr}(\Lambda^*A^*A\Lambda v) \leq \text{tr}(T^*B^2Tv) \leq \sup_\Lambda \text{tr}(\Lambda^*A^*A\Lambda v).$$

Thus, for some Λ , $\text{tr}(\Lambda^*A^*A\Lambda v) \leq \text{tr}(T^*B^2Tv)$. By (5–5),

$$\text{tr}(A^*Av) \leq C^2 \max\{\text{tr}(\Lambda^*A^*A\Lambda v), 1\} \leq C^2 \max\{\text{tr}(T^*B^2Tv), 1\},$$

which implies (5–7). □

Proposition 5.2 For $\alpha \in c_0$, the completely unconditional basis in $X^d(\alpha)$ is unique (up to permutative equivalence). More precisely: if (g_i) is a C -completely unconditional basis in $X^d(\alpha)$, then it is $16C^{11}$ -equivalent (up to a permutation) to the canonical basis in $X^d(\alpha)$.

Proof Let $(e_i(\alpha))$ be the canonical basis of $X^d(\alpha)$. Set $\beta_i = \|Ag_i\|$, and let $(e_i(\beta))$ the canonical basis of $X^d(\beta)$. By Proposition 5.1, the map $T : X^d(\alpha) \rightarrow X^d(\beta) : g_i \mapsto e_i(\beta)$ satisfies $\|T\|_{cb} \leq C$, $\|T^{-1}\|_{cb} \leq C^2$. By Theorem 4.8, $id : X^d(\beta) \rightarrow X^d(\alpha)$ satisfy $\|id\|_{cb}, \|id^{-1}\|_{cb} < 4C^4$. Thus, the operator $U = id \circ T$ is a complete isomorphism on $X^d(\alpha)$, with $Ug_i = e_i$, $\|U\|_{cb} < 4C^5$, and $\|U^{-1}\|_{cb} < 4C^6$. □

Corollary 5.3 Suppose $\alpha \in \mathcal{S}$, and Y is a C -completely complemented subspace of $X^d(\alpha)$. Then Y is 2^6C^2 -completely isomorphic to $X^d(\alpha')$, where α' is a subsequence of α .

Proof Let P be a projection from $X^d(\alpha)$ onto Y , with $\|P\|_{cb} \leq C$. Then $X^d(\alpha)$ is $2C$ -completely isomorphic to $Y \oplus Z$, where $Z = \ker P$. By Lemma 4.2, Y and Z are $\sqrt{2}$ -completely isomorphic to $X^d(\beta)$ and $X^d(\beta')$, respectively, where β and β' belong to \mathcal{S} . By Lemma 3.2(2), $X^d(\alpha)$ is $4C$ -completely isomorphic to $X^d(\gamma)$, where $\gamma = (\gamma_i) \in \mathcal{S}$ is the “join” of $\beta = (\beta_i)$ and $\beta' = (\beta'_i)$. More precisely, the sequence γ has the property that, for every $c \in [0, 1]$,

$$|\{i : \gamma_i = c\}| = |\{i : \beta_i = c\}| + |\{i : \beta'_i = c\}|.$$

Denoting the canonical basis of $X^d(\gamma)$ by $(e_i(\gamma))$, we see that $X^d(\beta) = \text{span}[e_i(\gamma) : i \in I]$, for some infinite set $I \subset \mathbb{N}$. By Theorem 4.8, the formal identity $id : X^d(\gamma) \rightarrow X^d(\alpha) : e_i(\gamma) \mapsto e_i(\alpha)$ satisfies $\|id\|_{cb}, \|id^{-1}\|_{cb} < 8C$. In particular, $X^d(\beta)$ 2^6C^2 -completely isomorphic to $\text{span}[e_i(\alpha) : i \in I]$. □

Remark 5.4 By [17], the completely unconditional basis in $\mathbf{R} \oplus \mathbf{C}$ is unique up to a permutation.

Remark 5.5 In general, the canonical basis of a subspace of $X^d(\alpha)$ ($\alpha \in \mathcal{S}$) need not be equivalent to a subsequence of the canonical basis of $X^d(\alpha)$. For instance, suppose the sequence $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ are defined by setting $\alpha_i = 2^{-n^2}$, $\beta_i = 2^{-n^2-n}$ for $4^{n^2} \leq i < 4^{(n+1)^2}$ ($n \in \{0\} \cup \mathbb{N}$). By Lemma 4.6, $X^d(\beta)$ embeds completely isomorphically into $X^d(\alpha)$. However, $(e_i(\beta))$ (the canonical basis of $X^d(\beta)$) is not equivalent to any subsequence of the canonical basis $(e_i(\alpha))$ of $X^d(\alpha)$. Indeed, suppose, for the sake of achieving a contradiction, that there exists a complete isomorphism T from $X^d(\beta)$ to a subspace of $X^d(\alpha)$, mapping $e_i(\beta)$ to $e_{k_i}(\alpha)$. Fix $n \in \mathbb{N}$ with $\max\{\|T\|_{cb}, \|T^{-1}\|_{cb}\} < 2^{n/2}$. Consider the sets

$$\begin{aligned} I_n &= \{4^{n^2} \leq i < 4^{(n+1)^2} : k_i < 4^{(n+1)^2}\}, \\ J_n &= \{4^{n^2} \leq i < 4^{(n+1)^2} : k_i \geq 4^{(n+1)^2}\}. \end{aligned}$$

By Pigeon-Hole Principle, with I_n or J_n has the cardinality grater than 4^{n^2+2n} . If $|I_n| > 4^{n^2+2n}$, consider

$$x = \sum_{i \in I_n} E_{i1} \otimes e_i(\beta) \in M_{4^{(n+1)^2}}(X^d(\beta))$$

(recall that E_{i1} is the ‘‘matrix unit’’ with 1 on the intersection of the first column and the i -th row, and zeroes everywhere else). By (4–2),

$$\|x\|_{M_{4^{(n+1)^2}}(X^d(\beta))}^2 = \max\{1, |I_n| \cdot 2^{-n^2-n}\} = |I_n| \cdot 2^{-n^2-n}.$$

However, by (4–2) again,

$$\|(M_{4^{(n+1)^2}} \otimes T)x\|_{M_{4^{(n+1)^2}}(X^d(\alpha))}^2 \geq \max\{1, |I_n| \cdot 2^{-n^2-n}\} = |I_n| \cdot 2^{-n^2},$$

yielding $\|T\|_{cb}^2 \geq 2^{-n}$. If $J_n > 4^{n^2+2n}$, consider

$$x = \sum_{i \in J_n} E_{i1} \otimes e_i(\beta) \in M_{4^{(n+1)^2}}(X^d(\beta)).$$

As before, $\|x\|^2 = |J_n| \cdot 2^{-n^2-n}$, while

$$\|(M_{4^{(n+1)^2}} \otimes T)x\|_{M_{4^{(n+1)^2}}(X^d(\alpha))}^2 \geq \max\{1, |J_n| \cdot 2^{-(n+1)^2}\} = |J_n| \cdot 2^{-(n+1)^2},$$

hence $\|T^{-1}\|_{cb}^2 \geq 2^{n+1}$. Thus, $\max\{\|T\|_{cb}, \|T^{-1}\|_{cb}\} \geq 2^{n/2}$, which yields the desired contradiction.

In certain situations, the canonical basis for every subspace of $X^d(\alpha)$ is equivalent to a subsequence of the canonical basis of $X^d(\alpha)$.

Theorem 5.6 For any $a > 1$ there exists $\alpha \in \mathcal{S} \setminus \ell_2$ such that any subspace Y of $X^d(\alpha)$ (finite or infinite dimensional) has a 1-completely unconditional basis, a -equivalent to a subsequence of the canonical basis of $X^d(\alpha)$.

Combining this result with Proposition 5.2, we obtain

Corollary 5.7 There exists $\alpha \in \mathcal{S} \setminus \ell_2$ such that any C -completely unconditional basic sequence in $X^d(\alpha)$ is AC^B -equivalent to a subsequence of the canonical basis of $X^d(\alpha)$ (here, A and B are positive).

Proof of Theorem 5.6 Assume $a < 2$. Pick a sequence of integers $1 = N_0 < N_1 < \dots$, s.t. $N_k > 2N_{k-1}$ for each k . Define a sequence $\alpha = (\alpha_i)$ by setting $\alpha_{2i} = a^{-k}$ for $N_k \leq i < N_{k+1}$, $\alpha_{2i-1} = 0$ for any $i \in \mathbb{N}$. Clearly, $\alpha \in c_0 \setminus \ell_2$. Let $A = \text{diag}(\alpha)$. For a subspace Y of $X(A)$, let $\beta = (\beta_i)$ is the sequence of singular values of $A|_Y$. Define the set \mathcal{I}_1 by setting $\mathcal{I}_1 = \{1, \dots, M\}$ if $\text{rank}(A|_Y) = M < \infty$, and $\mathcal{I}_1 = \mathbb{N}$ if $\text{rank}(A|_Y) = \infty$. We can also assume that the elements of $(\beta_i)_{i \in \mathcal{I}_1}$ are listed in the non-increasing order. Then $\beta_i \leq \alpha_{2i}$ for each i .

Denote the normalized eigenvectors of $(A|_Y)^*A|_Y$, corresponding to the eigenvalues β_i ($i \in \mathcal{I}_1$), by η_i . Furthermore, find the vectors $(\eta_i)_{i \in \mathcal{I}_0}$, forming an orthonormal basis in $\ker(A|_Y)$. For $i \in \mathcal{I}_0$, set $\beta_i = 0$. Let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_0$ (we assume that this union is disjoint). Then the family $(\eta_i)_{i \in \mathcal{I}}$ is the canonical basis for Y .

For each positive integer k , let M_k be the smallest value of i s.t. $\beta_i \leq a^{-k}$. Set $M_0 = 1$. In this notation, $a^{1-k} \geq \beta_i > a^{-k}$ iff $M_{k-1} \leq i < M_k$. As noted above, $\beta_i \leq \alpha_{2i}$, hence $M_k \leq N_k$. By our choice of the sequence (N_k) ,

$$M_k - M_{k-1} < M_k \leq N_k \leq N_{k+1} - N_k.$$

Thus, there exists an injective map $\pi : \mathcal{I} \rightarrow \mathbb{N}$ s.t. $\pi(\mathcal{I}_0) \subset \{2i - 1 : i \in \mathbb{N}\}$, and $\pi([M_{k-1}, M_k]) \subset \{2i : i \in [N_k, N_{k+1}]\}$ for each $k \in \mathbb{N}$. For $i \in \mathcal{I}_0$, $\alpha_{\pi(i)} = \beta_i = 0$, while for $i \in \mathcal{I}_1$, $a\alpha_{\pi(i)} = a^{1-k} \geq \beta_i > a^{-k} = \alpha_{\pi(i)}$. Define the operator $T : Y \rightarrow \text{span}[e_{\pi(i)} : i \in \mathcal{I}] \hookrightarrow X^d(\alpha)$, defined by $T\xi_i = e_{\pi(i)}$. By (4-1), T is a complete contraction, and $\|T^{-1}\|_{cb} \leq a$. \square

Remark 5.8 The proof of Theorem 5.6 shows that any subspace of $X^d(\alpha)$ is a -completely isomorphic to a completely contractively complemented subspace of $X^d(\alpha)$. Nevertheless, $X^d(\alpha)$ contains subspaces which are not completely complemented. To construct them, find a sequence (β_i) such that $1 \geq \beta_1 \geq \beta_2 \geq \dots > 0$, $\lim \beta_i = 0$, and furthermore, $\sum_i \gamma_i^2 = \infty$, where $\gamma_i = \alpha_i \beta_i$. Denote the canonical basis of $X^d(\alpha)$

by $(e_i)_{i \in \mathbb{N}}$. Then $e_{2i-1} = \xi_{2i-1} \oplus 0$, and $e_{2i} = \xi_{2i} \oplus \alpha_i \xi_{2i}$ ((ξ_j) is an orthonormal basis in ℓ_2). For $k \geq 0$ and $i \in [N_k, n_{k+1})$, let

$$(5-8) \quad f_i = \beta_i e_{2i} + \sqrt{1 - \beta_i^2} e_{2i-1} = (\beta_i \xi_{2i} + \sqrt{1 - \beta_i^2} \xi_{2i-1}) \oplus \gamma_i \xi_{2i}.$$

We show that Y is not completely complemented in $X^d(\alpha)$. Indeed, suppose, for the sake of contradiction, that there exists a c.b. projection P from $X^d(\alpha)$ onto Y . For $\varepsilon \in \{-1, 1\}^{\mathbb{N}}$, define an operator $\Lambda_\varepsilon \in B(X^d(\alpha))$ by setting $\lambda_\varepsilon e_j = \varepsilon_{\lfloor j/2 \rfloor} e_j$. For any such ε , (4-2) implies that Λ_ε is a complete isometry. For any $i \in \mathbb{N}$, we have $\Lambda_\varepsilon \xi = \varepsilon_i \xi$ whenever $\xi \in \text{span}[e_{2i-1}, e_{2i}]$. In particular, $\Lambda_\varepsilon f_i = \varepsilon_i f_i$. Let $Q = \text{Ave}_{\varepsilon \in \{-1, 1\}^{\mathbb{N}}} \Lambda_\varepsilon P \Lambda_\varepsilon$. Note that $\text{ran } Q \subset Y$, and $Q|_Y = I_Y$, hence Q is a projection onto Y . Furthermore, $\|Q\|_{cb} \leq \|P\|_{cb}$.

For each i , we have $Qe_{2i} = a_i f_i$, and $Qe_{2i-1} = b_i f_i$. The equations $Qf_i = f_i$ and (5-8) yield $a_i \beta_i + b_i \sqrt{1 - \beta_i^2} = 1$. Then $\sup_i \max\{|a_i|, |b_i|\} \leq \|Q\|$. As $\lim \beta_i = 0$, there exists $K \in \mathbb{N}$ such that $|b_i| > 1/2$ for $i > K$. Find $N \in \mathbb{N}$ s.t. $\sum_{i=K+1}^{K+N} \gamma_i^2 > 4\|Q\|_{cb}^2$ (this is possible, since $\sum_i \gamma_i^2 = \infty$). Consider $x = \sum_{i=K+1}^{K+N} E_{i1} \otimes e_{2i-1} \in M_N(X^d(\alpha))$. Then $\|x\| = 1$, and therefore,

$$\begin{aligned} \|Q\|_{cb} &\geq \|(I_{M_N} \otimes Q)x\| = \left\| \sum_{i=K+1}^{K+N} E_{i1} \otimes b_i f_i \right\| \\ &\geq \left\| \sum_{i=K+1}^{K+N} E_{i1} \otimes b_i \gamma_i \xi_{2i} \right\|_{M_N(\mathbf{C})} = \left(\sum_{i=K+1}^{K+N} |b_i|^2 \gamma_i^2 \right)^{1/2} \\ &\geq \frac{\left(\sum_{i=K+1}^{K+N} \gamma_i^2 \right)^{1/2}}{2} > \|Q\|_{cb}, \end{aligned}$$

which is impossible.

Remark 5.9 If any subspace of a Banach space E is complemented, then E is isomorphic to a Hilbert space. This was first proved in [14], see also [10] for sharper asymptotics of the isomorphism constants. A recent preprint [9] exhibits a class of separable Banach spaces E , not isomorphic to ℓ_2 , such that every subspace of E is isomorphic to a complemented subspace of E . It is not known whether E can be constructed in such a way that all of its subspaces have an unconditional basis.

6 Completely isomorphic classification of subspaces of $X(A)$

The main goal of this section is to prove Theorem 6.1 and Corollary 6.2 below. Recall that \mathfrak{C} is the set of compact contractions which are not Hilbert-Schmidt.

Theorem 6.1 *If $A \in B(\ell_2)$ belongs to \mathfrak{C} , then $(\mathbf{S}(X(A)), \simeq)$ is Borel bireducible to the complete \mathbf{K}_σ relation.*

Together with Corollary 4.5, this theorem immediately implies

Corollary 6.2 *If $A \in B(\ell_2)$ belongs to \mathfrak{C} , then the relation of complete biembeddability on $\mathbf{S}(X(A))$ is Borel bireducible to the complete \mathbf{K}_σ relation.*

The proof of Theorem 6.1 proceeds in two steps. First, we introduce the space S_A of sequences of non-negative generalized integers, with an equivalence relation $\overset{*}{\sim}$, and show the latter is Borel bireducible with $(\mathbf{S}(X(A)), \simeq)$. Then we prove that $(S_A, \overset{*}{\sim})$ is, in fact, a complete \mathbf{K}_σ relation.

Suppose $A \in B(\ell_2)$ is of class \mathfrak{C} . By Lemma 3.2, we can assume that $\|A\| < 1$, and $A \geq 0$. List the positive eigenvalues of A in the non-increasing order: $1 > \|A\| = s_1^o \geq s_2^o \geq \dots > 0$. In the terminology of Section 4, $(s_i^o)_{i \in \mathbb{N}} = \mathbf{D}(A)$. Clearly, $\lim_i s_i^o = 0$. Let $(\xi_i)_{i \in \mathbb{N}}$ be the normalized eigenvectors of A , corresponding to the eigenvalues s_i^o . We can identify $\text{span}[\xi_i : i \in \mathbb{N}]$ with ℓ_2 . Consider the operator $\hat{A} = \text{diag}(s_i^o) \oplus 0 \in B(\ell_2 \oplus \ell_2)$. By Corollary 3.8, $X(A) \simeq X(\hat{A})$. For the rest of this section, we assume that $A = \hat{A}$.

Denote by $(\xi'_i)_{i \in \mathbb{N}}$ the canonical orthonormal basis in the second copy of ℓ_2 . Then the canonical basis of $X(A)$ is the collection of vectors $e_i = \xi_i \oplus s_i^o \xi'_i$ and $f_i = \xi'_i \oplus 0$. As in (4-1), we have, for $n \times n$ matrices $a_1, b_1, a_2, b_2, \dots$,

$$(6-1) \quad \begin{aligned} & \left\| \sum_i a_i \otimes e_i + \sum_i b_i \otimes f_i \right\|_{M_n(X(A))}^2 \\ &= \max \left\{ \left\| \sum_i a_i a_i^* + \sum_i b_i b_i^* \right\|, \left\| \sum_i s_i^{o2} a_i^* a_i \right\| \right\}. \end{aligned}$$

For an infinite dimensional $Y \hookrightarrow X(A)$, we let $(s_i(Y)) = \mathbf{D}(A|_Y)$. Recalling the definition of \mathbf{D} from Section 4, we see that, if $\text{rank}(A|_Y) = \infty$, then $s_1(Y) \geq s_2(Y) \geq \dots$ are the positive singular values of $A|_Y$, listed in the non-increasing order. In the case of $\text{rank}(A|_Y) = n < \infty$, $s_1(Y) \geq s_2(Y) \geq \dots \geq s_n(Y)$ are the n positive singular values of $A|_Y$, and $s_i(Y) = 0$ for $i > n$. In this notation, $s_k^o = s_k(X(A))$. Clearly, $s_k(Y) \leq s_k^o$ for any k , and any $Y \hookrightarrow X(A)$.

For $k \in \mathbb{N}$, set $\mathbf{n}_k(Y) = \sup\{\ell \in \mathbb{N} : 2^{1-\ell} \geq s_k(Y)\}$. The sequence $\mathbf{n}(Y) = (\mathbf{n}_k(Y))_{k \in \mathbb{N}}$ belongs to $\mathbb{N}_*^{\mathbb{N}}$, where $\mathbb{N}_* = \mathbb{N} \cup \{\infty\}$ is viewed as the 1-point compactification of \mathbb{N} . For $k \in \mathbb{N}$ let $\alpha_k = \mathbf{n}_k(X(A))$. Define S_A as the set of all elements $\beta = (\beta_i)_{i \in \mathbb{N}} \in \mathbb{N}_*^{\mathbb{N}}$,

such that (1) $\beta_k \geq \alpha_k$ for any k , and (2) $\beta_1 \leq \beta_2 \leq \dots$. Equipping $\mathbb{N}_*^{\mathbb{N}}$ with its product topology, we see that S_A is closed.

For any infinite dimensional $Y \hookrightarrow X(A)$, the sequence $(\mathbf{n}_k(Y))_{k \in \mathbb{N}}$ belongs to S_A . Conversely, for any $\beta \in S_A$ there exists $Y \hookrightarrow X(A)$ s.t. $\beta = \mathbf{n}(Y)$. Indeed, suppose $\beta_k \in \mathbb{N}_*$, $\beta_k \geq \alpha_k$ for any k , and $\beta_1 \leq \beta_2 \leq \dots$. Let $g_i = \sin \phi_i e_i + \cos \phi_i f_i$, with $s_i^o \sin \phi_i = 2^{-\beta_i}$. We denote $\text{span}[g_i : i \in \mathbb{N}]$ by $\mathbf{Y}(\beta)$, where $\beta = (\beta_i)$. By (6-1), $\mathbf{n}(\mathbf{Y}(\beta)) = \beta$.

Define the relation $\overset{*}{\sim}$ on S_A as follows: $\beta \overset{*}{\sim} \gamma$ if there exists $K \in \mathbb{N}$ and $I \subset \mathbb{N}$ s.t. $|\beta_i - \gamma_i| \leq K$ for any $i \notin I$, and $\sum_{i \in I} (4^{-\beta_i} + 4^{-\gamma_i}) \leq K$. By Corollary 4.5, $\beta \overset{*}{\sim} \gamma$ iff $\mathbf{Y}(\beta) \simeq \mathbf{Y}(\gamma)$, and conversely, $Y \simeq Z$ iff $\mathbf{n}(Y) \overset{*}{\sim} \mathbf{n}(Z)$.

Proposition 6.3 \mathbf{n} and \mathbf{Y} are Borel maps.

This immediately yields:

Corollary 6.4 $(S_A, \overset{*}{\sim})$ and $(\mathbf{S}(X(A)), \simeq)$ are Borel bireducible to each other.

Proof of Proposition 6.3 First we handle the map \mathbf{Y} . We have to show that, for any open set $U \subset X(A)$, $\{\beta \in S_A : \mathbf{Y}(\beta) \cap U \neq \emptyset\}$ is Borel. But $\mathbf{Y}(\beta) \cap U \neq \emptyset$ iff there exist $m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m \in \mathbb{Q} + i\mathbb{Q}$ s.t. $\sum_{i=1}^m \lambda_i g_i \in U$. Here, the vectors (g_i) come from the definition of \mathbf{Y} . Note that, for each i , g_i depends solely (and continuously) on β_i . Therefore, for each m -tuple $(\lambda_i)_{i=1}^m$, $\sum_{i=1}^m \lambda_i g_i \in U$ is an open condition on β . Thus, $\{\beta \in S_A : \mathbf{Y}(\beta) \cap U \neq \emptyset\}$ is Borel.

Now consider \mathbf{n} . Fix $m, \beta_m \in \mathbb{N}$, and show that the set of all $Y \in \mathbf{S}(X(A))$, for which $\mathbf{n}_m(Y) > \beta_m \in \mathbb{N}$, is Borel. To this end, find a countable set \mathcal{O}_m of orthonormal m -tuples $\xi = (\xi_1, \dots, \xi_m)$ in $X(A)$, with the property that, for every $\varepsilon > 0$, and any orthonormal m -tuple (η_1, \dots, η_m) in $X(A)$, there exists $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_m$ s.t. $\|\xi_i - \eta_i\| < \varepsilon$ for any i . Furthermore, find a set Γ_m of m -tuples $\gamma = (\gamma_1, \dots, \gamma_m) \in \mathbb{C}^m$, dense in the unit sphere of ℓ_2^m .

The Minimax Principle (see e.g. [2, p. 75]) states that, for an operator $T \in B(H, K)$, we have $s_m(T) > b$ iff H has an m -dimensional subspace E such that $\|T\eta\| > b$ for any norm one $\eta \in E$. Therefore, $\mathbf{n}_m(Y) > \beta_m \in \mathbb{N}$ (that is, $s_m(A|_Y) \geq 2^{-\beta_m}$) iff for every $\varepsilon > 0$ there exists an m -tuple of orthonormal vectors $\eta_1, \dots, \eta_m \in Y$, s.t. $\|\sum_{i=1}^m \gamma_i A \eta_i\| > 2^{-\beta_m} - \varepsilon$ whenever $\sum_{i=1}^m |\gamma_i|^2 = 1$. This, in turn, is equivalent to the following statement: for every $r \in \mathbb{N}$, there exists $(\xi_1, \dots, \xi_m) \in \mathcal{O}_m$ s.t., for $1 \leq i \leq m$, $\text{Ba}(\xi_i, 1/r) \cap Y \neq \emptyset$ (here, $\text{Ba}(x, c)$ denotes the open ball of radius c , with the center at x), and $\|\sum_i \gamma_i A \xi_i\| > 2^{-\beta_m} - 1/r$ for every $(\gamma_i)_{i=1}^m \in \Gamma_m$. This condition is Borel, hence \mathbf{n} is Borel. \square

Lemma 6.5 $(S_A, \overset{*}{\sim})$ is a \mathbf{K}_σ relation.

Proof We have to show that the set $F = \{(\beta, \gamma) \in S_A \times S_A : \beta \overset{*}{\sim} \gamma\}$ is a \mathbf{K}_σ set, that is, a countable union of compact sets. To this end, define a family of subsets of $S_A \times S_A$, described below. For $K, n \in \mathbb{N}$ and $I_n \subset \{1, \dots, n\}$, define $F(K, n, I_n)$ as the set of all pairs (β, γ) ($\beta = (\beta_i), \gamma = (\gamma_i)$) with the property that $|\beta_i - \gamma_i| \leq K$ for $i \in \{1, \dots, n\} \setminus I_n$, and $\sum_{i \in I_n} (4^{-\beta_i} + 4^{-\gamma_i}) \leq K$. Let $F(K, n) = \cup_{I_n \subset \{1, \dots, n\}} F(K, n, I_n)$, and $F(K) = \cap_n F(K, n)$. It suffices to show that $F = \cup_{K \in \mathbb{N}} F(K)$. Indeed, $F(K, n, I_n)$ is a compact subset of $S_A \times S_A$, hence so is $F(K, n)$ (as a finite union of compact sets). Furthermore, $F(K)$ is also compact, and $\cup_K F(K)$ is \mathbf{K}_σ .

We show first that $F \subset \cup_K F(K)$. By definition, $\beta \overset{*}{\sim} \gamma$ if there exists $K \in \mathbb{N}$ and $I \subset \mathbb{N}$ s.t. $|\beta_i - \gamma_i| \leq K$ for any $i \notin I$, and $\sum_{i \in I} (4^{-\beta_i} + 4^{-\gamma_i}) \leq K$. Letting $I_n = I \cap \{1, \dots, n\}$, we see that $(\beta, \gamma) \in F(K, n, I_n)$ for each n , hence $(\beta, \gamma) \in F(K)$.

To prove the converse implication, suppose $(\beta, \gamma) \in F(K)$ for some K , and show that $\beta \overset{*}{\sim} \gamma$. Construct a tree $T \subset \{0, 1\}^{\mathbb{N}}$: for each n , $T \cap \{0, 1\}^n$ consists of all the sets I_n s.t. $|\beta_i - \gamma_i| \leq K$ for $i \notin I_n$, and $\sum_{i \in I_n} (4^{-\beta_i} + 4^{-\gamma_i}) \leq K$ (we identify the set of subsets of $\{1, \dots, n\}$ with $\{0, 1\}^n$). The set T is indeed a tree: if $I_n \in T$, then $I_n \cap \{1, \dots, m\} \in T$ for $m < n$. By assumption, T has arbitrarily long branches. By König's Lemma [11, p. 20], T has an infinite branch, which yields a set $I \subset \mathbb{N}$ s.t. $|\beta_i - \gamma_i| \leq K$ for $i \notin I$, and $\sum_{i \in I} (4^{-\beta_i} + 4^{-\gamma_i}) \leq K$. \square

Next consider a space $\Xi = \prod_{k \in \mathbb{N}} \Xi_k$, where $\Xi_k = \{0, \dots, k-1\}$, with the equivalence relation $c E_{\mathbf{K}_\sigma} b$ iff $\sup_i |c_i - b_i| < \infty$ (here, $c = (c_i)_{i \in \mathbb{N}}, b = (b_i)_{i \in \mathbb{N}}$). By [24], $E_{\mathbf{K}_\sigma}$ is a complete \mathbf{K}_σ relation, hence, by Lemma 6.5, it reduces $(S_A, \overset{*}{\sim})$. It remains to prove the converse.

Proposition 6.6 There exists a Borel map $\phi : \Xi \rightarrow S_A$ s.t. $\phi(b) \overset{*}{\sim} \phi(c)$ iff $c E_{\mathbf{K}_\sigma} b$.

Proof Recall that A is a compact contraction which is not Hilbert-Schmidt. Therefore, $\sum_i 4^{-\alpha_i} = \infty$, and $\lim_i \alpha_i = \infty$. Thus, there exists a sequence of positive integers $1 = p_0 < p_1 < p_2 < p_3 < \dots$ s.t. $\sum_{i \in I_k} 4^{-\alpha_i} > 4^{2k}$ ($I_k = [p_k, p_{k+1} - 1)$), and $\alpha_{p_{k+1}} > k + p_k$. Define $\phi((b_i)) = (b'_j)$ by setting $b'_j = \alpha_j + b_k$ if $j \in I_k$, and $b'_j = \min\{\alpha_j + k, \alpha_{p_{k+1}}\}$ if $p_k \leq j < p_{k+1}$. Clearly, ϕ is a Borel map. Moreover, if $c E_{\mathbf{K}_\sigma} b$, then $\phi(b) \overset{*}{\sim} \phi(c)$. Suppose, on the contrary, that $b' \overset{*}{\sim} c'$, where $b' = \phi(b)$ and $c' = \phi(c)$. Then there exists a set $I \subset \mathbb{N}$ and $K \in \mathbb{N}$ s.t. $|b'_j - c'_j| \leq K$ for $j \notin I$, and $\sum_{j \in I} (4^{-b'_j} + 4^{-c'_j}) \leq K$. We shall show that $|b_k - c_k| \leq K$ for all but finitely many

k 's. Indeed, otherwise there exist infinitely many k 's s.t. $I_k \subset I$ (this follows from the fact that $b'_j - c'_j = b_k - c_k$ for $j \in I_k$). But

$$\sum_{j \in I_k} (4^{-b'_j} + 4^{-c'_j}) \geq 2 \cdot 4^{-k} \sum_{j \in I_k} 4^{-\alpha'_j} > 1,$$

hence

$$\sum_{j \in I} (4^{-b'_j} + 4^{-c'_j}) \geq \sum_{I_k \subset I} \sum_{j \in I_k} (4^{-b'_j} + 4^{-c'_j}) = \infty,$$

a contradiction. □

Conclusion of the proof of Theorem 6.1 By Corollary 6.4, $(\mathbf{S}(X(A)), \simeq)$ and $(S_A, \overset{*}{\sim})$ are Borel bireducible to each other. By Lemmas 6.5 and 6.6, $(S_A, \overset{*}{\sim})$ is Borel bireducible to a complete \mathbf{K}_σ relation. □

Remark 6.7 For many separable Banach spaces X , it is known that the isomorphism relation on $\mathbf{S}(X)$ reduces certain “classical” relations, such as $E_{\mathbf{K}_\sigma}$ (see e.g. [1, 4, 5, 6]).

7 Proofs of Theorems 1.1, 1.2, and 1.3

Recall that the class \mathfrak{C} consists of all compact contractions, which are not Hilbert-Schmidt, and the family \mathfrak{F} is the set of all operator spaces $X(A)$, where $A \in B(\ell_2)$ belongs to \mathfrak{C} . Clearly, all these spaces are isometric to ℓ_2 .

Proof of Theorem 1.1 Suppose $X(A) \in \mathfrak{F}$. By Theorem 6.1 and Corollary 6.2, the relations of complete isomorphism and complete biembeddability on $\mathbf{S}(X(A))$ are complete \mathbf{K}_σ . To show that \mathfrak{F} contains a continuum of spaces, not completely isomorphic to each other, pick $A \in B(\ell_2) \cap \mathfrak{C}$. Consider a space $\Xi = \prod_{k \in \mathbb{N}} \Xi_k$, where $\Xi_k = \{0, \dots, k - 1\}$, with the equivalence relation $c E_{\mathbf{K}_\sigma} b$ iff $\sup_i |c_i - b_i| < \infty$ (here, $c = (c_i)_{i \in \mathbb{N}}$, $b = (b_i)_{i \in \mathbb{N}}$). By the results of Section 6, there exists a Borel map $\Phi : \Xi \rightarrow \mathbf{S}(X(A))$, such that $\Phi(b) \simeq \Phi(c)$ iff $b E_{\mathbf{K}_\sigma} c$. It remains to find a family $(b_\varepsilon)_{\varepsilon \in \{0,1\}^\mathbb{N}} \subset \Xi$, such that $b_\varepsilon E_{\mathbf{K}_\sigma} b_\delta$ iff $\varepsilon = \delta$. To this end, write \mathbb{N} as a disjoint union of infinite sets I_k ($k \in \mathbb{N}$). For any $\varepsilon = (\varepsilon_k)_{k=1}^\infty$, define

$$b_\varepsilon(i) = \begin{cases} 0 & i \in I_k, \varepsilon(k) = 0 \\ i - 1 & i \in I_k, \varepsilon(k) = 1 \end{cases}.$$

Clearly, this family (b_ε) has the desired properties. □

Proof of Theorem 1.2 Consider $A \in B(\ell_2)$ of class \mathfrak{C} . The existence of the canonical basis has been established at the beginning of Section 4, while its uniqueness follows from Proposition 5.2. \square

Proof of Theorem 1.3 Combine Theorem 1.2 with Theorem 5.6 and Corollary 5.7. \square

8 Isometric classification: proof of Theorem 1.4

We handle the real case. Begin by introducing a numerical invariant of subspaces of $X = \mathbb{R} \oplus_1 \ell_2$. Denote by P the “natural” projection onto \mathbb{R} . For $Y \in \mathbf{S}(X)$, define $c(Y) = \|P|_Y\|$.

Lemma 8.1 *For $Y \in \mathbf{S}(X)$, there exists $x \in Y$ such that $\|x\| = 1$, and $\|Px\| = c(Y)$. Moreover, if this x is written as $x = c(Y) \oplus (1 - c(Y))\xi_0$, then $Y = \text{span}[x, Y']$, where $Y' = \{0 \oplus \xi : \xi \in (Y \cap \ell_2) \cap \xi_0^\perp\}$.*

Proof If $c(Y) = 0$, the statement is trivial. Suppose $c(Y) = 1$. Then, for every $n \in \mathbb{N}$, there exists $t_n \in (1 - 1/n, 1]$ and $\xi_n \in \ell_2$ s.t. $\|\xi_n\| = 1$, and $t_n \oplus (1 - t_n)\xi_n \in X$. As Y is closed, $1 \oplus 0 \in Y$.

Next consider $c(Y) \in (0, 1)$. Suppose, for the sake of contradiction, that there is no x as in the statement of the lemma. Then for every $n \in \mathbb{N}$ there exist $t_n \in (c(Y) - 1/n, c(Y))$, and $\xi_n \in \ell_2$ s.t. $\|\xi_n\| = 1$, and $t_n \oplus (1 - t_n)\xi_n \in Y$. Passing to a subsequence if necessary, we can assume that (ξ_n) is a Cauchy sequence in ℓ_2 . Indeed, otherwise there exist $n_1 < n_2 < \dots$ and $\alpha > 0$, such that, for any i , $\|\xi_{n_{i+1}} - \xi_{n_i}\| > \alpha$. By the uniform convexity of Hilbert spaces (which follows, for instance, from the parallelogram identity), there exists $\beta > 0$ s.t. $\|(1 - t_{n_{i+1}})\xi_{n_{i+1}} + (1 - t_{n_i})\xi_{n_i}\|/2 < 1 - c(Y) - \beta$ for any i . Define

$$y_i = \frac{t_{n_{i+1}} + t_{n_i}}{2} \oplus \frac{(1 - t_{n_{i+1}})\xi_{n_{i+1}} + (1 - t_{n_i})\xi_{n_i}}{2} \in Y.$$

Then $\|y_i\| < 1 - \beta$, and $\lim_i \|Py_i\| = c(Y)$. Therefore, $\|P|_Y\| > c(Y)$, which is impossible.

Thus, the sequence (ξ_n) converges to some $\xi_0 \in \ell_2$. Then $x = c(Y) \oplus (1 - c(Y))\xi_0$ is the limit of the sequence $t_n \oplus (1 - t_n)\xi_n$, hence it belongs to Y . Clearly, $\|Px\| = c(Y)$, and $Y = \text{span}[x, Y \cap \ell_2]$. Moreover, any $\xi \in Y \cap \ell_2$ is orthogonal to ξ_0 . Indeed,

otherwise there exists $\xi \in Y \cap \ell_2$ and $z \in \mathbb{C}$ s.t. $\|\xi_0 + z\xi\| < \|\xi_0\|$. Then $x' = x + (0 \oplus \xi) = c(Y) \oplus (\xi_0 + z\xi)$ belongs to Y , $\|x'\| < 1$, and $\|Px'\| = c(Y)$, which is impossible. \square

For $t \in [0, 1]$, define $\phi(t) = t + \sqrt{(1-t)^2 + 1}$. Clearly, ϕ is continuous and increasing.

Lemma 8.2 For $Y \in \mathbf{S}(X)$,

$$\phi(c(Y)) = \sup \{ \liminf_i \|x + y_i\| : x, y_i \in Y, \|x\| = \|y_i\| = 1, y_i \xrightarrow{w} 0 \}.$$

Moreover, there exist a norm 1 $x \in Y$, and a normalized weakly null sequence (y_i) in Y , such that $\phi(c(Y)) = \|x + y_i\|$ for every i .

Proof Assume $c(Y) \in (0, 1)$ (only minimal changes are needed to handle $c(Y) \in \{0, 1\}$). Write $x = t \oplus (1-t)\xi$ and $y_i = t_i \oplus (1-t_i)\xi_i$. Here, $t, t_i \in [0, 1]$, $\xi_i \in \ell_2$, and $\|\xi_i\| = 1$. As $y_i \rightarrow 0$ weakly, $t_i \rightarrow 0$, and $\langle \xi, \xi_i \rangle \rightarrow 0$. Therefore, $\lim_i \|x + y_i\| = \phi(t)$. Taking the supremum over all $x \in Y$, we prove the desired equality. Furthermore, by Lemma 8.1, $Y = \text{span}[x, Y \cap \ell_2]$, where $x = c(Y) \oplus (1 - c(Y))\xi_0$, $\xi_0 \in \ell_2$ has norm 1, and $Y \cap \ell_2$ is orthogonal to ξ_0 . Let (ξ_i) be an orthonormal basis in $Y \cap \ell_2$. Then $\phi(c(Y)) = \|x + y_i\|$ for every i , and $y_i \xrightarrow{w} 0$. \square

Lemma 8.3 If Y and Z are infinite dimensional subspaces of X , and Y is almost isometrically embeddable into Z , then $c(Y) \leq c(Z)$.

Proof By definition, for every $\lambda \in (1, 1.1)$, there exist a subspace $W \hookrightarrow Z$ and a contraction $T : Y \rightarrow W$ with $\|T^{-1}\| < \lambda$. It suffices to show that

$$(8-1) \quad \phi(c(W)) \geq \lambda^{-1}\phi(c(Y)) - 2(\lambda - 1).$$

Indeed, then we would conclude

$$\phi(c(Z)) \geq \phi(c(W)) \geq \lambda^{-1}\phi(c(Y)) - 2(\lambda - 1).$$

As the above inequality holds for any $\lambda > 1$, we conclude that $\phi(c(Y)) \leq \phi(c(Z))$. By the monotonicity of ϕ , $c(Y) \leq c(Z)$.

By Lemma 8.2, there exists a normalized weakly null sequence (y_i) in Y , and a norm one $x \in Y$, such that $\phi(c(Y)) = \|x + y_i\|$. In the space W , consider the elements $x' = Tx/\|Tx\|$, and $y'_i = Ty_i/\|Ty_i\|$. Then the sequence (y'_i) is weakly null, and

$$(8-2) \quad \|x' + y'_i\| \geq \|T(x + y_i)\| - \|(1 - \|Tx\|^{-1})Tx\| - \|(1 - \|Ty_i\|^{-1})Ty_i\|.$$

But $\|T^{-1}\| \|T(x + y_i)\| \geq \|x + y_i\|$, hence $\|T(x + y_i)\| > \lambda^{-1}\phi(c(Y))$. Furthermore, $1 \geq \|Tx\| > \lambda^{-1}$, hence $\|(1 - \|Tx\|^{-1})Tx\| < \lambda - 1$. Similarly, $\|(1 - \|Ty_i\|^{-1})Ty_i\| < \lambda - 1$. By (8-2), $\|x' + y'_i\| > \lambda^{-1}\phi(c(Y)) - 2(\lambda - 1)$. Applying Lemma 8.2, we obtain (8-1). \square

Lemma 8.4 *If Y and Z are infinite dimensional subspaces of X , and $c(Y) = c(Z)$, then Y is isometric to Z .*

Proof We consider the case $c(Y) = c(Z) \in (0, 1)$ (the extreme cases of $c(Y) = c(Z) \in \{0, 1\}$ are handled similarly). By Lemma 8.1, Y contains a norm one $y = c(Y) \oplus \xi_Y \in Y$ (note that $\|\xi_Y\| = 1 - c(Y)$), s.t. $Y = \text{span}[y, Y \cap \ell_2]$, and $Y' = Y \cap \ell_2$ is orthogonal to ξ_Y . Thus, for any $\xi \in Y'$, $\|y + \xi\| = c(Y) + \sqrt{(1 - c(Y))^2 + \|\xi\|^2}$. Similarly, $Z = \text{span}[z, Z']$, and $\|z + \eta\| = c(Z) + \sqrt{(1 - c(Z))^2 + \|\eta\|^2}$ for any $\eta \in Z'$. As Y' and Z' are both infinite dimensional separable Hilbert spaces, there exists an isometry T' from Y' onto Z' . We complete the proof by defining the isometry T from Y onto Z by setting $Ty = z$, and $T|_{Y'} = T'$. \square

Proof of Theorem 1.4 By Lemmas 8.3 and 8.4, the following statements are equivalent for $Y, Z \in \mathbf{S}(X)$: (i) Y and Z are isometric, (ii) $d(Y, Z) = 1$, (iii) Y and Z are isometrically bi-embeddable, (iv) Y and Z are almost isometrically bi-embeddable, (v) $c(Y) = c(Z)$. Denote that canonical basis for ℓ_2 by e_0, e_1, \dots , and consider a map

$$\Phi : [0, 1] \rightarrow \mathbf{S}(X) : t \rightarrow \text{span}[t \oplus (1 - t)e_0, 0 \oplus e_1, 0 \oplus e_2, \dots].$$

Then $c(\Phi(t)) = t$, hence $\Phi(t_1)$ and $\Phi(t_2)$ satisfy any (equivalently, all) of the relations (i) – (iv) iff $t_1 = t_2$. It remains to prove that the maps Φ and c are Borel.

To handle Φ , consider an open ball $U \subset X$ with the center at $\alpha \oplus \sum_{i=0}^N \beta_i e_i$ and radius r . Then $\Phi(t) \cap U \neq \emptyset$ iff there exist $\lambda_0, \dots, \lambda_N \in \mathbb{Q}$ s.t.

$$|\alpha - t\lambda_0| + \left(|\beta_0 - (1 - t)\lambda_0|^2 + \sum_{i=1}^N |\beta_i - \lambda_i|^2 \right)^{1/2} < r.$$

This inequality describes a Borel subset of $[0, 1]$. As any open subset of X is a countable union of open balls, the map Φ is Borel.

To deal with c , consider the sets

$$U_t = \{s \oplus \xi \in \mathbb{R} \oplus_1 \ell_2 : |s| > t, \|\xi\| < \sqrt{1 - t^2}\}$$

($t \in [0, 1]$). Clearly, U_t is an open subset of X , and $c(Y) > t$ iff $Y \cap U_t \neq \emptyset$. \square

Remark 8.5 Theorem 1.4 holds not only for $\mathbb{R} \oplus_1 \ell_2$, but also for $\mathbb{R} \oplus_p \ell_2$, for $1 \leq p < 2$.

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